Risk Measures and Optimal Reinsurance

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis, we study the optimal reinsurance design problem and extend the classical model in three different directions:

1. In the first framework, we add the additional assumption that the reinsurer can default on its obligations. If the indemnity is beyond the reinsurer's payment ability, the reinsurer fails to pay for the exceeding part and this induces a default risk for the insurer. In our model, the reinsurer is assumed to measure the risk of an insured loss by Value-at-Risk regulation and prepares the same amount of money as the initial reserve. As soon as the indemnity is larger than this value plus the premium, default occurs. From the insurer's point of view, two optimization problems are going to be considered when the insurer: 1) maximizes his expectation of utility; 2) minimizes the VaR of his retained loss.

2. In the second framework, the reinsurance buyer (insurer) adopts a convex risk measure ρ to control his total loss while the reinsurance seller (reinsurer) price the reinsurance contract by Wang's premium principle with distortion g. Without specifying a particular convex risk measure ρ and distortion g, we obtain a general expression for the optimal reinsurance contract that minimizes the insurer's total risk exposure.

3. In the third framework, we study optimal reinsurance designs from the perspectives of both an insurer and a reinsurer and take into account both an insurer's aims and a reinsurer's goals in reinsurance contract designs. We develop optimal reinsurance contracts that minimize the convex combination of the VaR risk measures of the insurer's loss and the reinsurer's loss under two types of constraints, respectively. The constraints describe the interest of both the insurer and the reinsurer. With the first type of constraints, the insurer and the reinsurer each have their limit on the VaR of their own loss. With the second type of constraints, the insurer has a limit on the VaR of his loss while the reinsurer has a target on his profit from selling a reinsurance contract. For both types of constraints, we derive the optimal reinsurance form for a wide class of reinsurance policies and under the expected value reinsurance premium principle.

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To my beloved.

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Chapter 1

Introduction

1.1 Optimal (Re-)Insurance Problem

A reinsurance contract, bought by the insurer from the reinsurer to protect against the insurer's potential aggregate claim, is an important risk-sharing tool for the insurer and became a popular research area in both academic research and industry use. The fast increase in demand for reinsurance can be explained by changes in the insurance market. Previously, insurance companies used to assume independence between risks. It led to the belief that the aggregate reimbursement could be predicted by using the sample mean as long as there are enough risks, and thus the insurer thought it could well control his risk exposure and capital requirements. However, it became more common to see the same risk scenarios influencing the issued policies, for example based on events such as natural catastrophes, terrorism, and financial crises. Under these situations, the aggregate reimbursement is hard to predict and the insurer may face extraordinary losses, which might lead to the insurer's bankruptcy. By signing a reinsurance contract the insurer can transfer part of his risk to the reinsurer, who can diversify the large-scale risks. The insurer can benefit from the reinsurance contract as a way of stabilizing the volatility of its balance sheet and, at the same time, increasing its capacity to afford more business or risk. Therefore, the reinsurance contract, which provides a mechanism for risk sharing and diversification, becomes an effective risk management tool for the insurer.

Let us denote by X the underlying (aggregate) risk faced by the insurer. Conventionally, X is assumed to be a non-negative random variable. Under a reinsurance contract, the reinsurer agrees to pay indemnity $I(X)$ to the insurer and requires a premium P_I. The premium principle is selected by the reinsurer and this will be discussed in more detail in Section 1.3. Thus, when a loss $X = x$ occurs, $I(x)$ is the part ceded to the reinsurer and the insurer will only need to cover the retained loss $x - I(x)$. The function $I(x)$ is commonly described as *compensation function*, *indemnification function*, or *ceded loss function*, while $R(x) \triangleq x - I(x)$ is known as *retained loss function*. We shall denoted by $\mathcal{I}$ the pool of all available reinsurance contracts. The total loss faced by the insurer with a reinsurance contract now becomes $X - I(X) + P_I$. If we denote by W_0 the initial wealth of the insurer, after receiving indemnity from the reinsurer, the insurer's terminal wealth is $W_0 - X + I(X) - P_I$. Essentially, the insurer may want to choose a reinsurance contract to maximize his expected utility of terminal wealth corresponding to a utility function $v(\cdot)$, namely,

$$\max_I \mathbb{E}\left[v\left(W_0 - X + I(X) - P_I\right)\right]; \tag{1.1}$$

or to have the smallest potential risk under a particular risk measure ρ, namely,

$$\min_I \rho\left(X - I(X) - P_I\right). \tag{1.2}$$

Optimization problems (1.1) and(1.2) can be conducted among a pool of reinsurance contracts. Before giving more detailed explanations about the optimal reinsurance problem, it is necessary to define a "*feasible*" reinsurance contract.

In the design of the reinsurance contract, moral hazard is an important issue that needs to be avoided. Essentially, the reinsurer needs to minimize the audit cost and wants to avoid paying more reimbursement to the insurer because of the manipulation of the actual loss by the insurer. Since the reinsurance is a risk sharing mechanism between the insurer and the reinsurer, an essential principle is "the higher the risk occurs, the more loss for both risk-sharing parties", otherwise, there exists moral hazard. If the ceded loss function $I(x)$ is not a non-increasing function, say there exists $x < y$ such that $I(x) > I(y)$, even if the loss reaches the level y, the insurer may partially hide this actual loss and only report loss x to the reinsurer to receive a higher compensation from the reinsurer. The adverse but similar situation is when the retained loss function $R(x)$ is not a non-decreasing function, because the insurer could then pay less to the policyholder by inflating the claim to the reinsurer. As a result, the reinsurance market commonly accepts the reinsurance contract, which has both non-decreasing and continuous ceded and retained loss functions on the support of the loss X. Denote by $\bar{X}$ the essential supremum of the random variable X, i.e.

$$\bar{X} = \operatorname{ess\,sup} X = \inf\left\{a \in \mathbb{R} : \mathbb{P}\left(X > a\right) = 0\right\},$$

and $\bar{X} \in \mathbb{R}^+ \cup \{+\infty\}$ could be a finite or infinite value. In the remainder of this thesis, we will call an insurance/reinsurance contract I as *"feasible"* with respect to X if I satisfies the following conditions:

1. $I : \left[0, \bar{X}\right] \setminus \{\infty\} \to \left[0, \bar{X}\right] \setminus \{\infty\}$ such that $I(0) = 0$ and I is non-decreasing;
2. $|I(y) - I(x)| \le |y - x|$, for any non-negative x and y.

The second condition is also known as 1-Lipschitz continuous condition or "slow growing" property. It is easy to check that the ceded loss function is feasible, i.e. $I \in \mathcal{I}$, is equivalent to that the retained loss function is feasible, i.e. $R \in \mathcal{I}$. Throughout this thesis, we denote by $\mathcal{I}$ the set of all feasible insurance/reinsurance contracts with respect to the given loss X and all optimization will be conducted over $\mathcal{I}$. For any $I \in \mathcal{I}$, I is continuous and non-decreasing, thus I is differentiable almost everywhere. We denote by I' the right derivative of I. That is,

$$I'(x) \triangleq \lim_{y \downarrow x} \frac{I(y) - I(x)}{y - x}, \quad \text{for any } x \ge 0.$$

Clearly, for any $I \in \mathcal{I}$, its right derivative I' is a right continuous function satisfies

$$I'(x) \in [0, 1], \text{ and } I(x) = \int_0^x I'(t)\,\mathrm{d}t \text{ for all } x \ge 0.$$

The reinsurance model could be classified into the "insurer-reinsurer-oriented" model and "insurer-oriented" models. In the one-period "insurer-oriented" model, [Arrow, 1963] and [Arrow, 1971] provided the fundamental work on the optimal insurance design problem. Arrow has shown that the stop-loss insurance treaty is the optimal solution to the following maximization problem:

$$\begin{aligned} &\max_{I \in \mathcal{I}} \mathbb{E}\left[v\left(W_0 - X + I(X) - p\right)\right], \\ &\text{such that } P_I = \mathbb{E}[I(X)] = p. \end{aligned} \tag{1.3}$$

This is a particular case of Problem (1.1) when the premium is calculated by the expected premium principle and fixed equal to p. The utility function $v(\cdot)$ is commonly assumed to be an increasing concave function and this represents the risk-averse bearing of the insured who is seeking risk sharing. When the initial wealth is non-random, say $W_0 = w_0$ for some constant w_0, by using $u(x) \triangleq -v(w_0 - x - p) + v(w_0 - p)$, Problem (1.3) is equivalent to the following minimization problem

$$\min_{I \in \mathcal{I}} \mathbb{E}[u(X - I(X))],$$

where $u(\cdot)$ is a non-negative, increasing and convex function. As a special case of Problem (1.3), the variance minimization model, considered by [Bowers et al., 1997], [Kaas et al., 2001] and [Gerber, 1979], is another widely used insurance model. In this model, the insured wants to minimize the variance of his retained loss:

$$\min_{I \in \mathcal{I}} Var(X - I(X)).$$

If the premium is still determined by the expected value of the ceded loss, the stop-loss insurance treaty is the optimal choice for the insured. In these fundamental models, the concave utility describes the insurer's risk-averse bearing which may be induced by the view that the insured is not able to diversify risk, and thus is seeking for ceding risk to another market participant; while the expected premium reflects the fact that the insurer is risk-neutral and he could diversify the risk.

Along with the reinsurance market's development, the fundamental model considered by Arrow has been extended in many directions subject to different objective functions/criteria or premium principle, or with a relaxation of constraints on feasible insurances, or with some additional constraints. [Deprez and Gerber, 1985] replaced the expected premium by the convex and Gateaux differentiable premium principle and released the budget constraint on the premium and obtained a sufficient and necessary condition for the optimal insurance contract. [Wang, 1996] and [Wang et al., 1997] proposed a list of natural axioms, which suggests that a "sound" premium price should be a Choquet integral of the indemnity which is convex but not Gateaux differentiable. As an application, [Young, 1999] extended the work of [Deprez and Gerber, 1985] to Wang's premium principle which will be introduced in Section 1.3. Under the variance minimization model, [Gajek and Zagrodny, 2000] used the standard deviation premium principle and assumed that the insurance treaty is acceptable to the insured as long as its premium does not exceed an upper budget constraint; while [Kusuoka, 2001] adopted the mean-variance principle for the premium and derived an explicit form for the optimal reinsurance contract. Kaluszka also considered the same premium principle under the utility maximization models and explored the optimal solution corresponding to a specific utility function, see [Kaluszka, 2004]. [Kaluszka, 2005] generalized his previous result to an even more general framework: convex risk measure with convex premium principle.

More recently, the optimal reinsurance decision problem has been revisited under different risk measures. [Cai and Tan, 2007] introduced the general risk measures Value-at-Risk (VaR) and Conditional Tail Expectation (CTE) into the reinsurance's model and sought for the optimal stop-loss contracts and optimal quota-share contracts under various premium principles. [Cai et al., 2008] also considered the extension of the previous works when all reinsurances with non-decreasing convex indemnities are regarded as feasible. [Cheung, 2010] extended their results under Wang's premium principle and

[Cheung et al., 2014] resolved the optimal reinsurance problem under more general convex risk measure subject to the expected premium principle.

In all the aforementioned theoretical studies, however, only the structure of the optimal reinsurance with a single reinsurer is studied. In practice, reinsurance is an effective risk-sharing tool between the insurer and the reinsurer and it is common that a few reinsurers, say N reinsurers, with heterogeneous preferences could participate in one reinsurance treaty. The insurer may pay less cost for ceding an amount of loss by formulating a competitive reinsurance portfolio. It should be noted that, diversifying between different reinsurers is never optimal for the insured when all reinsurers are risk neutral, i.e. for $i = 1, \ldots, N$, insurer i adopts actuarial price principle $(1+\theta_i)\mathbb{E}[I_i(X)]$ with risk loading $\theta_i \geq 0$ to price an indemnity I_i. It is because the premium is increasing linearly and the insurer can always get the cheapest reinsurance from the reinsurer with the smallest risk loading and use it against the total indemnity $I = \sum_{i=1}^{N} I_i$. However, when reinsurers are risk averse, to cover the same amount of additional unit of X, the higher the level of X the more the marginal premium asked by an reinsurer. Even if, for example, Reinsurer 1 provides cheaper insurance than Reinsurer 2 when X is small, his rate of marginal premium will increase and become higher than that of Reinsurer 2 eventually, therefore it is optimal for the insurer to buy insurance from Reinsurer 2 against the high-level portion of X. The detailed argument for this can be found in Proposition 4.2 of [Malamud et al., 2012]

- If all reinsurers are risk neutral, then only the reinsurer with the smallest discount factor will participate in a trade;
- If reinsurers are risk averse and $\bar{X}$ is sufficiently large, then all reinsurers will participate in a trade.

To the best of our knowledge though, [Malamud et al., 2012] firstly analysis the optimal risk sharing problem in the presence of more than two agents, or equivalently, optimal insurance design with multiple insurers. In their model, insured and all insurers are intertemporal expected utility maximizer with different von Neumann-Morgenstern utilities and discount factors. Thus, their results on the optimal insurance design can be viewed as an extension of [Raviv, 1979]. A more recent work about optimal reinsurance problems with multiple reinsurers is given by [**?**]. They take VaR and CVaR risk measures as criteria and seek to reduce the risk exposure of an insurer under the assumption that one reinsurer adopts the expected value principle while the second reinsurer's premium principle belongs to a general class with three basic axioms: distribution invariance, risk loading and preserving stop-loss order. The premium principle for the second reinsurer is very flexible in the sense that it contains eight of eleven commonly used premium principles listed in [Wang et al., 1997]. They conclude that over both the VaR and CVaR risk measures criteria, an optimal reinsurance arrangement for an insurer is to cede two adjacent layers $(I_1(X), I_2(X))$ defined as follows:

$$I_1(x) \triangleq (x-d_1)^+ - (x-d_2)^+ \text{ and } I_2(x) \triangleq (x-d_2)^+ - (x-d_3)^+,$$

for $0 \leq d_1 \leq d_2 \leq d_3$, and I_i is distributed to Reinsurer i.

Another interesting extension of the classical optimal reinsurance model is adding the counterparty risk as a background risk. Counterparty risk, also called credit risk or default risk, recently became a popular topic in the optimal reinsurance design problem. When the reinsurance buyer has a big loss, the reinsurance seller may only be able to pay part of the promised insurance indemnity, instead of the entire amount. It implies a default risk for the reinsurance buyer. [Cummins and Danzon, 1997] and [Cummins et al., 2002] discussed the importance of insolvency risk in insurance markets. More recently, the impact of counterparty risk on the optimal sharing transfers has captured more attention, see [Biffis and Millossovich, 2012], [Bernard and Ludkovski, 2012] and [Asimit et al., 2013].

In the "insurer-reinsurer-oriented" model, it is similar to a game-theoretic problem that reflects both insurer and reinsurer's interest. As the two parties of a reinsurance contract, an insurer and a reinsurer have conflicting interests. An optimal form of reinsurance from one party's point of view may be not acceptable to the other party as pointed out by [Borch, 1960]. To illustrate this conflict, consider one example when both the insurer and the reinsurer use VaR to measure their own risk. From the insurer's perspective, the insurer prefers to buy a reinsurance contract that is a solution to the optimization problem

$$\min_{I \in \mathcal{I}} \mathrm{VaR}_\alpha \left(X - I(X) + P_I\right). \tag{1.4}$$

However, from the reinsurer's point of view, the reinsurer likes to sell a reinsurance contract that is a solution to the optimization problem

$$\min_{I \in \mathcal{I}} \mathrm{VaR}_\beta \left(I(X) - P_I\right), \tag{1.5}$$

where α and β are the VaR risk levels of the insurer and the reinsurer, respectively. Optimal solutions to Problems (1.4) and (1.5) are different. Indeed, when the reinsurance premium P_I is determined by the expected value principle, namely $P_I = (1+\theta)\mathbb{E}[I(X)]$ with a positive risk loading factor $\theta > 0$, [Cheung et al., 2014] proved that the optimal reinsurance form for Problem (1.4) is

$$I_i^*(x) = (x - \mathrm{VaR}_{\frac{1}{1+\theta}}(X))^+ - (x - \mathrm{VaR}_\alpha(X))^+.$$

Then, using the solution to Problem (1.4), it is easy to obtain that the optimal reinsurance form for Problem (1.5) or for the reinsurer is

$$I_r^*(x) = x - (x - \mathrm{VaR}_{\frac{1}{1+\theta}}(X))^+ + (x - \mathrm{VaR}_\beta(X))^+.$$

Obviously, $I_i^* \neq I_r^*$ almost everywhere, and thus in Problems (1.4) and (1.5), the optimal reinsurance form for one party is not optimal for the other. Indeed, the optimal contract minimizing the VaR of one party's loss may lead to an unacceptable large value for the VaR of the other party's loss.

Hence, a very interesting question is to take into consideration both an insurer's objectives and a reinsurer's goals in optimal reinsurance design so that an optimal reinsurance form is acceptable to both parties. There are two general ways to consider both an insurer's objectives and a reinsurer's goals in an optimal reinsurance design. One way is to minimize or maximize an objective function that considers both an insurer's aims and a reinsurer's goals, and the other way is to minimize or maximize an objective function from one party's point of view under some constraints on the other party's goals and on the party's own objectives. [Borch, 1960] first addressed this issue by discussing the quota-share and stop-loss reinsurance contracts and deriving the optimal retention of these contracts under the optimization criterion of maximizing the product of the expected utility functions of the two parties' terminal wealth. Recently, [Hürlimann, 2011] has readdressed this issue by studying the combined quota-share and stop-loss contracts and obtaining the optimal retention of these contracts under the optimization criterion of minimizing the sum of the variances of the losses of the insurer and the reinsurer and several other related optimization criteria. [Cai et al., 2013] proposed the optimization criteria of maximizing the joint survival probability and the joint profitable probability of the two parties and derived sufficient conditions for a reinsurance contract to be optimal in a wide class of reinsurance policies and under a general reinsurance premium principle. Using the results of [Cai et al., 2013] , [Fang and Qu, 2014] derived the optimal retentions of a combined quota-share and stop-loss reinsurance under the criterion of maximizing the joint survival probability of the two parties under the expected value reinsurance premium principle.

In this thesis, we are going to design the optimal reinsurance contract in several more general frameworks. In Chapter 2, we add the assumption that the reinsurer may fail to fulfill his liability when the incurred loss exceeds his maximal payment ability. We will investigate this default impact on the form of the optimal reinsurance. With the presence of default risk, in most cases, the optimal reinsurance has a limited stop form but requires a lower deductible or an extra deductible in the middle. In Chapter 3, we adopt the convex law-invariant risk measure and Wang's premium principle which will be introduced in Section 1.2 and Section 1.3 respectively. Both risk measure or premium principle are represented in a general form rather than a particular expression. We will discuss the general form of the optimal reinsurance contract within this framework and the general solution is consistent with the existing result when the risk measure or the premium principle is specifically identified. In Chapter 4, we minimize the convex combination of the VaR risk measures of the insurer's loss and the reinsurer's loss subject to two sets of the insurer's and the reinsurer's constraints. Each set of constraints includes restrictions, according to risk management concern or profit concern, from both the insurer and the reinsurer. Therefore, the optimal reinsurance can be acceptable by both parties. In all of these three optimal reinsurance models, the choices of the risk measure and the premium principle influence the form of the optimal reinsurance policy. In the next two sections of this chapter, we are going to introduce risk measures and premium principles and some important results in the existing literature.

1.2 Risk Measure

In risk management, quantifying the risk of a financial position is a key task that gives rise to extensive discussions both from a theoretical and a practical point of view. In a financial market, let Ω denote the set of all possible scenarios, and the future value of a financial position can be described by a random variable X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A real value is assigned to each financial position to represent its risk level. Such functionals are called risk measures.

Definition 1.2.1 *A measure of risk ρ is a mapping from a set of risk random variables $\mathcal{X}$ into the real value line.*

A risk measure ρ could be an arbitrary functional on $\mathcal{X}$ which is the set of all random variables X on $(\Omega, \mathcal{F}, \mathbb{P})$. However, in practice, a risk measure is expected to satisfy certain conditions. In the sequel, we use the positive part of X to represent the loss and the negative part of X to represent the gain.

Definition 1.2.2 *A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is called a monetary risk measure if ρ satisfies the following conditions for all $X, Y \in \mathcal{X}$.*

1. *Monotonicity: If $X \leq Y$, then $\rho(X) \leq \rho(Y)$.*

2. *Translation invariance: For any $m \in \mathbb{R}$, $\rho(X + m) = \rho(X) + m$.*

For a monetary risk measure ρ, the number $\rho(X)$ can be used as a capital requirement, that is as the minimal extra cash which should be added to the financial position X to make it acceptable. Thus, a monetary measure of risk ρ induces the acceptance set.

Definition 1.2.3 *The acceptance set associated to a risk measure ρ is the set denoted by $\mathcal{A}_\rho$ and defined by*

$$\mathcal{A}_\rho \triangleq \{X \in \mathcal{X} : \rho(X) \leq 0\} .$$

Conversely, one can make a list, denoted by $\mathcal{A}$, of all acceptable financial positions according to an investor's own preference and define a risk measure $\rho_{\mathcal{A}}$ associated with $\mathcal{A}$.

Definition 1.2.4 *The risk measure associated to an acceptance set $\mathcal{A}$ is defined by*

$$\rho_{\mathcal{A}}(X) \triangleq \inf \{m : X - m \in \mathcal{A}\} .$$

The discussion about the acceptance set and desirable properties of a risk measure was initiated in the coherent case by [Artzner et al., 1999].

As one of the most popular measures of risk, Value-at-Risk (VaR) has achieved the highest status of being written into industry regulation.

Definition 1.2.5 *The Value-at-Risk (VaR) of random variable X at level α is defined as the lower α-quantile of X*

$$\mathrm{VaR}_\alpha(X) \triangleq \inf \{x \geq 0 : S_X(x) \leq \alpha\} ,$$

where S_X is the survival function of X.

Value-at-Risk is a monetary risk measure, however, it is heavily criticized for not being subadditive and does not take into account the severity of an incurred damage event. As a response to these deficiencies, the notion of coherent risk measures was firstly introduced in [Artzner et al., 1997] and further developed in [Artzner et al., 1999]. The authors discussed methods of measurement of (market and non-market) risks and proposed a set of four desirable properties.

Definition 1.2.6 *If the set Ω of all possible scenarios is finite. A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is called a coherent risk measure if the following axioms are satisfied: for any X, $Y \in \mathcal{X}$:*

1. *Monotonicity: If $X \leq Y$, then $\rho(X) \leq \rho(Y)$.*

2. *Translation invariance: For any $m \in \mathbb{R}$, $\rho(X+m) = \rho(X) + m$.*

3. *Subadditivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$.*

4. *Positive homogeneity: For any $\lambda \geq 0$, $\rho(\lambda X) = \lambda \rho(X)$.*

Clearly, a coherent risk measure is a monetary measure satisfying subadditivity and positive homogeneity.

Proposition 1.2.1 *A risk measure ρ is coherent if and only if there exists a family $\mathcal{P}$ of probability measures on the set of states of nature, such that*

$$\rho(X) = \sup \{\mathbb{E}_{\mathbb{P}}[-X] : \mathbb{P} \in \mathcal{P}\}. \tag{1.6}$$

Proposition 1.2.1, which was provided in [Artzner et al., 1999], gave a general representation for all coherent risk measures in terms of generalized scenarios; and the same result, in a different context, has been also obtained in [Huber, 1981]. Using this representation result, a specific coherent measure – worst conditional expectation (WCE) – was suggested and WCE is shown to be, under some assumptions, the least expensive among coherent risk measures. It is accepted by regulators since it is more conservative than the VaR measurement.

Definition 1.2.7 (Worst Conditional Expectation at level α)

$$WCE_\alpha(X) \triangleq \sup \{\mathbb{E}[X|A] : \mathbb{P}(A) > \alpha\}.$$

This notion is closely related to the Conditional Tail Expectation (CTE), which is defined as follows, but does not coincide with WCE in general.

Definition 1.2.8 (Conditional Tail Expectation at level α)

$$CTE_\alpha(X) \triangleq \mathbb{E}[X|X > VaR_\alpha(X)].$$

WCE is in fact coherent but useful only in a theoretical setting since it requires the knowledge of the whole underlying probability space, while CTE lends itself naturally to practical applications but it is not coherent.

Since then, many scholars have made various important contributions along this direction. The definition of coherent risk measure was extended by [Delbaen, 2000] to arbitrary probability spaces, which were assumed to be finite probability spaces in [Artzner et al., 1999]. In Definition 1.2.6, [Artzner et al., 1999] use $\mathcal{X} = \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$, which is the set of all bounded random variables on a finite atom-less probability space $(\Omega, \mathcal{F}, \mathbb{P})$. [Delbaen, 2000] suggested that, when defining the coherent risk measure on the space of all real valued random variables, the value of a coherent risk measure may by $+\infty$ and when this happens, it means that the risk is very bad and is unacceptable for the economic agent, or something like a risk that cannot be insured. Meanwhile, on a separable metric space Ω which may not be finite, a coherent risk measure ρ has representation (1.6) is equivalent to the Fatou property: i.e.

$$\rho(X) \leq \liminf_{n\to\infty} \rho(X_n), \text{ wheneve } \sup_n \|X_n\|_\infty < \infty \text{ and } X_n \xrightarrow{\mathbf{P}} X,$$

where $\xrightarrow{\mathbf{P}}$ denotes convergence in probability.

As a typical example of coherent risk measure, Expected Shortfall (ES), also known as Average Value-at-Risk (AVaR), makes up for several drawbacks that VaR has and serves as an important risk measure in insurance and credit risk management.

Definition 1.2.9 (Expected Shortfall/Average Value-at-Risk at level α)

$$\mathrm{AVaR}_\alpha(X) \triangleq \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\xi(X)\,\mathrm{d}\xi.$$

AVaR was explored in [Acerbi and Tasche, 2002], in which the authors presented four characterizations : 1) as integral of all the quantiles below the corresponding level; 2) as limit in a tail strong law of large numbers; 3) as minimum of a certain function; 4) as maximum of WCEs when the underlying probability space varies. In this way, they showed that AVaR is a coherent risk measure and easy to compute and to estimate and therefore is complementary and even in some aspects superior to the other notions.

The notion of convex risk measures was introduced in [Follmer and Schied, 2002] as a generalization of coherent risk measures.

Definition 1.2.10 *A risk measure $\rho : \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is called a convex risk measure if the following axioms are satisfied for any X, $Y \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$:*

1. *Monotonicity: If $X \leq Y$, then $\rho(X) \leq \rho(Y)$.*

2. *Translation invariance: For any $m \in \mathbb{R}$, $\rho(X+m) = \rho(X) + m$.*

3. *Convexity: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ for any $\lambda \in [0,1]$.*

Since in many situations the risk of a position might increase in a nonlinear way with the size of the position, they suggested to relax the conditions of positive homogeneity and of subadditivity and to require, instead of these two properties, the convexity property, which is a weaker condition. They also provide a corresponding extension of the representation theorem in terms of probability measures on the underlying space of scenarios. This representation theorem works for a general probability space and the space of all bounded random variables. A risk measure satisfies the *law-invariant* property if it assigns the same value to two risky positions having a common distribution. If one imposes the law-invariant property as an additional axiom, the representation result was obtained by [Kusuoka, 2001] in the coherent case, and by [Feirrwlli and Rosazza Gianin, 2005] in the convex case. In [Jouini et al., 2006], the authors have shown that a law-invariant convex risk measure ρ satisfies the Fatou property. Moreover, the authors gave the following useful representation result for law-invariant convex risk measures:

Lemma 1.2.2 *Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is an atomless probability space. Denote $\mathcal{P}([0,1])$ to be the set of all Borel probability measures on $[0,1]$. For a function $\rho : \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$, the following are equivalent:*

1) *ρ is a law invariant convex risk measure.*

2) *There is a function $\beta : \mathcal{P}([0,1]) \to [0,\infty]$ which is law invariant, lower semi-continuous and convex such that*

$$\rho(Y) = \sup_{\mu \in \mathcal{P}([0,1])} \left(\int_0^1 \mathrm{AVaR}_\alpha(Y)\mu(\mathrm{d}\alpha) - \beta(\mu) \right),$$

where AVaR_α is given by Definition 1.2.9.

1.3 Premium Principle

A premium principle is a rule for assigning a premium to an insurance risk under a particular indemnity. It could also be viewed as a risk measure that measures the ceded loss for the reinsurer and thus the choice of the reinsurance premium principle essentially reflects the reinsurer's preferences and hedging strategy. There is a lot of discussion on the axioms that should be satisfied by a premium principle in order to make sure the premium is fair enough to the market. The following gives a list of some reasonable properties of a premium principle P, where $P : \mathcal{X} \to [0, \infty]$ is a real-valued functional on the set of all risk random variables.

- P1 Independence: $P(X)$ depends only on the cumulative distribution function of X.
- P2 Risk loading: $P(X) \geq \mathbb{E}[X]$.
- P3 No unjustified risk loading: If a risk X is identically equal to a constant $c \geq 0$ almost everywhere, then $P(X) = c$.
- P4 No rip-off: $P(X) \leq \operatorname{ess\,sup} X$ for all risk X.
- P5 Translation invariance: $P(X + a) = P(X) + a$ for all X and all $a \geq 0$.
- P6 Scale invariance: $P(bX) = bP(X)$ for all X and all $b \geq 0$.
- P7 Additivity: $P(X + Y) = P(X) + P(Y)$ for all X and Y.
- P8 Subadditivity: $P(X + Y) \leq P(X) + P(Y)$ for all X and Y.
- P9 Supperadditivity: $P(X + Y) \geq P(X) + P(Y)$ for all X and Y.
- P10 Independent additivity: $P(X+Y) = P(X)+P(Y)$ for all X and Y are independent.
- P11 Comonotonic additivity: $P(X+Y) = P(X)+P(Y)$ for all X and Y are comonotonic.
- P12 Monotonicity: If $X \leq Y$ with probability 1, then $P(X) \leq P(Y)$.
- P13 Continuity: $\lim_{a\to 0^+} P(\max\{X - a, 0\}) = P(X)$, and $\lim_{a\to\infty} P(\min\{X, a\}) = P(X)$.

A reasonable premium principle also needs to preserve the ordering between two random losses. Here, we provide two common definitions of ordering between two risk variables X and Y:

Definition 1.3.1 (First stochastic dominance ordering "$\preceq_{ST}$") *A random variable X is smaller than another random variable Y in the first stochastic dominance ordering, denoted by $X \preceq_{ST} Y$ if*

$$\mathbb{E}[\psi(X)] \leq \mathbb{E}[\psi(Y)], \quad \text{for any non-decreasing function } \psi,$$

provided the expectation exists.

Definition 1.3.2 (Stop-loss function) *A stop-loss function ψ with deductible $d > 0$ has the form: $\psi(x) = (x - d)^+$.*

Definition 1.3.3 (Stop-loss ordering "$\preceq_{SL}$")

$$X \preceq_{SL} Y, \text{ if } \mathbb{E}[\psi(X)] \leq \mathbb{E}[\psi(Y)] \text{ for any stop-loss function } \psi.$$

P14 Preserves first stochastic dominance ordering: If $X \preceq_{ST} Y$, then $P(X) \leq P(Y)$.

P15 Preserves stop-loss ordering: If $X \preceq_{SL} Y$, then $P(X) \leq P(Y)$.

When the insurer accepts many independent risks, the sample mean and the theoretical mean for insurance indemnities become closer and the aggregate reimbursement can be predicted by using the law of large number. Therefore, premiums can be determined by using the expected value plus a positive risk loading, which is used to cover all expense as well as a profit return. For example, the Expected Value Premium Principle is defined as $P(X) = (1 + \theta)\mathbb{E}[X]$, where $\theta > 0$ is the risk loading. The Expected Value Premium Principle satisfies all properties listed above except P4 "No rip-off".

There are some other ways to determine a premium principle. An actuary can first list properties that he wants the premium principle to satisfy and then find an appropriate one. Or, he can adopt a particular economic theory and then determine the resulting premium principle. A widely used list of axioms for a premium principle in a competitive market, where the insurance prices are determined by the collective efforts of all buyers and sellers, was proposed by [Wang et al., 1997]. They suggested that an appropriate insurance premium principle should satisfy properties P1, P11, P12 and P15 and this kind of premium principle is called as *Wang's premium principle*. An important concept in Wang's premium principle is that of "*distortion function*" which is defined as follows:

Definition 1.3.4 (Distortion) *Let $\mathbb{P}$ be a probability measure on a σ-algebra Ω. For an increasing function g on $[0, 1]$ with $g(0) = 0$ and $g(1) = 1$, the function $g \circ \mathbb{P}$ is called a **distorted** probability and the function g is called a **distortion function**.*

[Wang et al., 1997] showed that, under some assumptions, if the market premium functional $P : \mathcal{X} \to [0, \infty]$ satisfies these four properties, then there is a unique distortion function g such that

$$P(X) = P(1) \int X \, \mathrm{d}(g \circ \mathbb{P}) = P(1) \int_0^\infty g \circ S_X(t) \mathrm{d}t.$$

In particular, if $P(1) = 1$, then P has a Choquet integral representation:

$$P(X) = \int_0^\infty g \circ S_X(t) \mathrm{d}t,$$

If, furthermore, g is concave, then P preserves the stop-loss ordering. Finally, this result is essentially the extension of Yaari's Representation Theorem [Yaari, 1987] to all unbounded random variables.

Chapter 2

Counterparty Default Risk with VaR-Regulated Initial Reserve

In this chapter, we consider the impact of the default risk on the optimal reinsurance design. In most studies on optimal reinsurance, one assumes that a reinsurer will pay the promised loss $I(X)$ regardless of its solvency or equivalently, one ignores the potential default by a reinsurer. Indeed, default risk can be reduced if a reinsurer has a sufficiently large initial capital or reserve. However, default might occur even if the initial capital of a reinsurer is very large. In a reinsurance contract I, a reinsurer may fail to pay the promised amount $I(X)$ or a reinsurer may default due to different reasons. One of the main reasons could be that the promised amount $I(X)$ exceeds the reinsurer's solvency. The larger is the initial reserve of a reinsurer, the smaller is the likelihood that default will occur. This is why the initial capital of a seller (reinsurer) of a reinsurance contract should meet some requirements by regulation to reduce default risk.

In this chapter, we propose a reinsurance model with regulatory initial capital and default risk. We assume that the initial capital or reserve of a seller (reinsurer) of a reinsurance contract I is determined through regulation by the Value-at-Risk (VaR) of its promised indemnity $I(X)$, and denote the initial capital of the reinsurer by $\omega_I = \mathrm{VaR}_\alpha(I(X))$, where $\mathrm{VaR}_\alpha(Z) = \inf\{z : \Pr\{Z > z\} \leq \alpha\}$ is the VaR of a random variable Z and $0 < \alpha < 1$ is called the risk level. Usually, α is a small value such as $\alpha = 0.01$ or 0.05. We assume that the reinsurer charges a reinsurance premium P_I based on the promised indemnity $I(X)$. The insurer is aware of the potential default by the reinsurer but the worst case for the insurer is that the reinsurer only pays $\omega_I + P_I$ if $I(X) > \omega_I + P_I$. Thus, when the insurer is seeking for optimal reinsurance strategies and taking account of the potential default by the reinsurer, the insurer assumes the worst indemnity $I(X) \wedge (\omega_I + P_I)$

from the reinsurer. Indeed, when $\omega_I = \mathrm{VaR}_\alpha(I(X))$, we know $\Pr(I(X) > \omega_I + P_I) \leq \alpha$ or the probability of default by the reinsurer is not greater than the value α, which could be an acceptable risk level for the insurer. Hence, under the proposed reinsurance model, the total retained risk or cost of the insurer is $X - I(X) \wedge (\omega_I + P_I) + P_I$ and the insurer's terminal wealth is $w_0 - X + I(X) \wedge (\omega_I + P_I) - P_I$, where w_0 is the initial capital of the insurer.

To avoid tedious discussions and arguments, in this chapter, we assume that the survival function $S_X(x)$ of the underlying loss random variable X is continuous and strictly decreasing on $(0, \infty)$ with $0 < S_X(0) \leq 1$. The survival function $S_X(x)$ has a possible jump at point zero which means it is possible that no claim raised from the policyholder to the insurer. This assumption has also been used in [Cheung, 2010]. Furthermore, we assume that $P_I = (1+\theta)\mathbb{E}[I(X)]$, i.e., the reinsurance premium is determined by the expected value principle, where $\theta > 0$.

As discussed in the introduction, utility maximization and risk measure minimization are two main optimization problems in optimal reinsurance design. In this chapter, we consider these two problems separately under the default assumption. We firstly investigate the optimal reinsurance contract when the insurer wants to maximize his utility of the terminal wealth. Secondly, we consider the risk measure minimization problem when the insurer uses VaR as well. All proofs are given in Section 2.4. Results in this chapter can also be found in [Cai et al., 2014].

2.1 Utility Maximization

In this section, we assume that the insurer wants to determine an optimal reinsurance strategy I^* that maximizes the expected utility of its terminal wealth of $w_0 - X + I(X) \wedge (\omega_I + P_I) - P_I$ under an increasing concave utility function v. That is, we study the following optimization problem:

$$\begin{aligned} &\max_{I \in \mathcal{I}} \mathbb{E}\left[v\left(w_0 - X + I(X) \wedge (\omega_I + P_I) - P_I\right)\right] \\ &\text{such that } P_I = (1+\theta)\mathbb{E}[I(X)] = p, \end{aligned} \tag{2.1}$$

where $0 < p \leq (1+\theta)\mathbb{E}(X)$ is a given reinsurance premium budget for the insurer. This optimal reinsurance problem can be viewed as the extension of the classical optimal reinsurance problem without default risk, which was first studied by [Arrow, 1963] and [Borch, 1960]. As illustrated later in the chapter, as $\alpha \to 0$, Problem (2.1) is reduced to the classical optimal reinsurance problem without default risk studied by [Arrow, 1963] and [Borch, 1960].

We can also recover the solutions of [Arrow, 1963] and [Borch, 1960] from our solution to Problem (2.1).

First, we point out that by taking $u(x) = -v(w_0 - p - x)$, Problem (2.1) is equivalent to the following minimization problem:

$$\begin{aligned} &\min_{I \in \mathcal{I}} \mathbb{E}\Big[u\Big(X - I(X) \wedge (\omega_I + P_I)\Big)\Big] \\ &\text{such that } P_I = (1+\theta)\mathbb{E}[I(X)] = p, \end{aligned} \tag{2.2}$$

where u is an increasing convex function. Throughout this section, we assume $\mathbb{E}|u^{(k)}(X)| < \infty$ for $k = 0, 1, 2$, and all expectation exists and integration and differentiation are exchangeable by assuming sufficient regularity conditions.

Second, we notice that for any $I \in \mathcal{I}$, the function $I(x)$ is continuous on $[0, \infty)$. In addition, for any $0 \le x < y$, if $I(y) = I(x) + y - x$, then $I(t) = t - (x - I(x))$ on the interval $[x, y]$.

For any fixed premium budget $0 < p \le (1+\theta)\mathbb{E}[X]$, we denote the set of all feasible contracts with the given reinsurance premium p by

$$\mathcal{I}_p = \{I \in \mathcal{I} : P_I = (1+\theta)\mathbb{E}[I(X)] = p\}.$$

Note that if $p = (1+\theta)\mathbb{E}[X]$, then $\mathcal{I}_p = \{I(x) \equiv x\}$, which contains only one reinsurance contract $I(x) \equiv x$, and thus Problem (2.2) reduces to the trivial case. Hence, throughout this section, we assume $p \in \big(0, (1+\theta)\mathbb{E}[X]\big)$. Then Problem (2.2) can be written as

$$\min_{I \in \mathcal{I}_p} \mathbb{E}\Big[u\Big(X - I(X) \wedge (\omega_I + P_I)\Big)\Big] = \min_{I \in \mathcal{I}_p} H(I), \tag{2.3}$$

where

$$H(I) \triangleq \mathbb{E}\Big[u\Big(X - I(X) \wedge (\omega_I + P_I)\Big)\Big].$$

To solve the infinite-dimensional optimization Problem (2.3), we first show that for any given reinsurance contract $I \in \mathcal{I}_p$, there exists a contract $k_I \in \mathcal{I}_p$ such that $H(k_I) \le H(I)$ and k_I is determined by four variables. Thus, we can reduce the infinite-dimensional optimization Problem (2.3) to a finite-dimensional optimization problem. To do so, we recall the definition of convex order.

Definition 2.1.1 *Random variable X is said to be smaller than random variable Y in convex order, denoted as $X \preccurlyeq_{cx} Y$, if $\mathbb{E}[u(X)] \le \mathbb{E}[u(Y)]$ for any convex function $u(\cdot)$ such that the expectations exist.*

Since $u(\cdot)$ is a convex function, for each I, we want to construct a contract $k_I \in \mathcal{I}_p$ satisfying $\tilde{k}_I(X) \preccurlyeq_{cx} \tilde{I}(X)$. The following lemma was given by [Ohlin, 1969] and it provides a useful criterion for the convex order.

Lemma 2.1.1 *Let X be a random variable, h_1 and h_2 be increasing functions such that $\mathbb{E}[h_1(X)] \leq \mathbb{E}[h_2(X)]$. If there exists $x_0 \in \mathbb{R} \cup \{+\infty\}$ such that $h_1(x) \geq h_2(x)$ for all $x < x_0$ and $h_1(x) \leq h_2(x)$ for all $x > x_0$, then $h_1(X) \preccurlyeq_{cx} h_2(X)$.*

Throughout this paper, we adopt the following notation

$$(a)^+ \triangleq \max\{a, 0\}, \; a \wedge b \triangleq \min\{a, b\}, \text{ and } a \vee b \triangleq \max\{a, b\}.$$

The following lemma shows that for any given reinsurance contract $I \in \mathcal{I}_p$, there exists a contract $k_I \in \mathcal{I}_p$ such that $H(k_I) \leq H(I)$.

Theorem 2.1.2 *Denote $a = \mathrm{VaR}_\alpha(X)$. For any $I \in \mathcal{I}_p$, there exists $k_I \in \mathcal{I}_p$ such that $H(k_I) \leq H(I)$ and k_I has the form*

$$k_I(x) = (x - d_1)^+ - (x - (d_1 + I(a)))^+ + (x - d_2)^+ - (x - (d_2 + p))^+ + (x - d_3)^+ \quad (2.4)$$

for some $(d_1, d_2, d_3) \in \mathbb{R}_+^3$ satisfies $0 \leq d_1 \leq d_1 + I(a) \leq a \leq d_2 < d_2 + p \leq d_3 \leq \infty$.

Remark 2.1.1 *We point out that for any $I \in \mathcal{I}$, I is continuous and non-decreasing. Thus, $\omega_I = \mathrm{VaR}_\alpha(I(X)) = I(\mathrm{VaR}_\alpha(X)) = I(a)$. As $\alpha \to 0$ or $a \to \infty$, we have $I(a) \to I(\infty)$ and hence $I \wedge (\omega_I + P_I) = I$. In other words, as $\alpha \to 0$ or $a \to \infty$, Problem (2.2) is reduced to the following classical problem without the default risk:*

$$\begin{aligned} &\min_{I \in \mathcal{I}} \mathbb{E}\left[u\left(X - I(X)\right)\right] \\ &\textit{such that } P_I = (1+\theta)\mathbb{E}[I(X)] = p. \end{aligned} \quad (2.5)$$

The optimal solution to Problem (2.5), given in [Arrow, 1963] and [Borch, 1960], is a stop-loss reinsurance $I_0^*(x) = (x - d^*)^+$, where d^* is uniquely determined by the premium condition. This classical result can also be recovered from Lemma 2.1.2. Indeed, as $a \to \infty$, the ceded loss function k_I in (2.4) is reduced to the form

$$k_I(x) = (x - d_1)^+ - (x - d_1 - I(\infty))^+,$$

for some $0 \leq d_1 \leq \infty$ determined by the premium condition $(1+\theta)\mathbb{E}[k_I(X)] = p$. If $I(\infty) = \infty$, then $k_I(x) = (x - d^*)^+$. If $I(\infty) < \infty$, it is easy to see that $d_1 < d^*$ and $x - (x - d^*)^+$ crosses $x - k_I(x)$ at most once from above on $[0, \infty)$. Thus, by Lemma 2.1.1, we have $X - (X - d^*)^+ \preccurlyeq_{cx} X - k_I(X)$. Thus, $\mathbb{E}\left[u\left(X - (X - d^*)^+\right)\right] \leq \mathbb{E}\left[u\left(X - k_I(X)\right)\right]$ for any k_I with $I(\infty) < \infty$. Therefore, in either $I(\infty) = \infty$ or $I(\infty) < \infty$, $X - (X - d^*)^+ \preccurlyeq_{cx} X - k_I(X) \preccurlyeq_{cx} X - I(X)$. Thus, $(x - d^*)^+$ is the optimal solution to Problem (2.5).

In the rest of this chapter, we assume $0 < \alpha < S_X(0)$ and thus $0 < a < \infty$. Otherwise, the initial reserve is $\omega_I = \mathrm{VaR}_\alpha(X) = 0$ which has no meaning in the practice. Theorem 2.1.2 reduces the infinite dimension of Problem (2.3) to a finite dimension problem. To see that, we denote

$$\mathcal{I}_{p,0} \triangleq \{I \in \mathcal{I}_p \text{ and } I \text{ having the expression (2.4)}\}.$$

Then, thanks to Theorem 2.1.2, we see that Problem (2.3) is equivalent to the following minimization problem

$$\min_{I\in\mathcal{I}_{p,0}} \mathbb{E}\Big[u\Big(X - I(X) \wedge (\omega_I + P_I)\Big)\Big] = \min_{I\in\mathcal{I}_{p,0}} H(I). \tag{2.6}$$

It is still not easy to solve Problem (2.6) since it involves the four variables d_1, d_2, d_3 and $I(a)$. To solve Problem (2.6), we first need to discuss the properties of the set $\mathcal{I}_{p,0}$.

For any given $\xi \in [0, a]$, define contract $I_{0,\xi} \in \mathcal{I}$ as

$$I_{0,\xi}(x) = (x - a + \xi)^+ - (x - a)^+ \tag{2.7}$$

and denote the reinsurance premium based on $I_{0,\xi}$ by $p_{0,\xi}$, that is

$$p_{0,\xi} = (1+\theta)\mathbb{E}\left[I_{0,\xi}(X)\right] = (1+\theta)\int_{a-\xi}^{a} S_X(x)\mathrm{d}x. \tag{2.8}$$

Furthermore, define contract $I_{M,\xi} \in \mathcal{I}$ as

$$I_{M,\xi}(x) = x - (x - \xi)^+ + (x - a)^+ \tag{2.9}$$

and denote the reinsurance premium based on $I_{M,\xi}$ by $p_{M,\xi}$, that is

$$p_{M,\xi} = (1+\theta)\mathbb{E}\left[I_{M,\xi}(X)\right] = (1+\theta)\left(\int_0^{\xi} + \int_a^{\infty}\right) S_X(x)\mathrm{d}x. \tag{2.10}$$

Denote

$$\xi_0 = \inf\left\{\xi \in [0, a] : p_{M,\xi} \geq p\right\}, \tag{2.11}$$

$$\xi_M = \sup\left\{\xi \in [0, a] : p_{0,\xi} \leq p\right\}. \tag{2.12}$$

Conventionally, $\xi_0 = 0$ if the set $\{\xi \in [0, a] : p_{M,\xi} \geq p\}$ is empty; $\xi_M = a$ if the set $\{\xi \in [0, a] : p_{0,\xi} \leq p\}$ is empty. Note that p is assumed to satisfy $0 < p < (1+\theta)\mathbb{E}[X]$, that is $p_{0,0} = 0 < p < (1+\theta)\mathbb{E}[X] = p_{M,a}$. Obviously, both $p_{0,\xi}$ and $p_{M,\xi}$ are continuous and increasing in $\xi \in [0, a]$. Therefore, $p_{M,\xi_0} = p_{M,0} > p$ if $p_{M,0} > p$, and $p_{M,\xi_0} = p$ if $p_{M,0} \leq p$; meanwhile, $p_{0,\xi_M} = p_{0,a} < p$ if $p_{0,a} < p$ and $p_{0,\xi_M} = p$ if $p_{0,a} \geq p$.

Lemma 2.1.3 *For any $\xi \in [0, a]$, the following three conditions are equivalent:*

(1) $\{I \in \mathcal{I}_{p,0} : I(a) = \xi\} \neq \emptyset$,

(2) $p_{0,\xi} \leq p \leq p_{M,\xi}$,

(3) $\xi \in [\xi_0,\, \xi_M]$.

Moreover, the set $\mathcal{I}_{p,0}$ can be written as the union of disjoint non-empty sets, namely

$$\mathcal{I}_{p,0} = \bigcup_{\xi_0 \leq \xi \leq \xi_M} \{I \in \mathcal{I}_{p,0} : I(a) = \xi\}\,.$$

It will be proved in Theorem 2.1.6 that Problem (2.6) is equivalent to the following two-step minimization problem:

$$\min_{0\leq\xi\leq a} \left\{ \min_{I\in\mathcal{I}_{p,0},\, I(a)=\xi} H(I) \right\} = \min_{\xi_0\leq\xi\leq\xi_M} \left\{ \min_{I\in\mathcal{I}_{p,0},\, I(a)=\xi} H(I) \right\} = \min_{\xi_0\leq\xi\leq\xi_M} H(I^*_\xi), \tag{2.13}$$

where $I^*_\xi = \arg\min_{I\in\mathcal{I}_{p,0},\, I(a)=\xi} H(I)$ for any given $\xi \in [\xi_0,\, \xi_M]$.

To derive the expression of the minimizer I^*_ξ of (2.13), we define contract $I_{1,\xi}(x) \in \mathcal{I}$ as

$$I_{1,\xi}(x) = (x - a + \xi)^+ - (x - a - p)^+ \tag{2.14}$$

and denote the reinsurance premium based on $I_{1,\xi}$ by $p_{1,\xi}$, that is

$$p_{1,\xi} = (1+\theta)\mathbb{E}\left[I_{1,\xi}(X)\right] = (1+\theta)\int_{a-\xi}^{a+p} S_X(x)\mathrm{d}x. \tag{2.15}$$

Furthermore, we define contract $I_{2,\xi}(x) \in \mathcal{I}$ as

$$I_{2,\xi}(x) = x - (x-\xi)^+ + (x-a)^+ - (x-a-p)^+ \tag{2.16}$$

and denote the reinsurance premium based on $I_{2,\xi}$ by $p_{2,\xi}$, that is

$$p_{2,\xi} = (1+\theta)\mathbb{E}\left[I_{2,\xi}(X)\right] = (1+\theta)\left(\int_0^\xi + \int_a^{a+p}\right) S_X(x)\mathrm{d}x. \tag{2.17}$$

In addition, throughout this paper, we denote

$$\xi_1 = \begin{cases} \sup\left\{\xi \in [\xi_0, \xi_M] :\, p_{2,\xi} < p\right\}, & \text{if} \quad p_{2,\xi_0} < p, \\ \xi_0, & \text{if} \quad p_{2,\xi_0} \geq p, \end{cases} \tag{2.18}$$

and

$$\xi_2 = \begin{cases} \inf\left\{\xi \in [\xi_0, \xi_M] : \, p_{1,\xi} > p\right\}, & \text{if} \quad p_{1,\xi_M} > p, \\ \xi_M, & \text{if} \quad p_{1,\xi_M} \leq p. \end{cases} \tag{2.19}$$

It is not hard to check by the definitions of ξ_1 and ξ_2, that $\xi_0 \leq \xi_1 \leq \xi_2 \leq \xi_M$ and that at least one of the three inequalities is strict.

Lemma 2.1.4 *The set $[\xi_0,\, \xi_M]$ has only the following three possible partitions: (1) if $\xi_0 = \xi_1$, then $\xi_0 = \xi_1 = \xi_2 = 0 < \xi_M$ and $[\xi_0,\, \xi_M] = [0,\, \xi_M]$; (2) if $\xi_2 = \xi_M$, then $\xi_0 < \xi_1 = \xi_2 = \xi_M = a$ and $[\xi_0,\, \xi_M] = [\xi_0,\, a]$; and (3) if $\xi_0 < \xi_1$ and $\xi_2 < \xi_M$, then $\xi_0 < \xi_1 < \xi_2 < \xi_M$ and $[\xi_0,\, \xi_M] = [\xi_0,\, \xi_1] \cup [\xi_1,\, \xi_2] \cup [\xi_2,\, \xi_M]$.*

Now, in the following lemma, for any given $\xi \in [\xi_0,\, \xi_M]$, we solve the inner minimization problem $\min_{I \in \mathcal{I}_{p,0},\, I(a)=\xi} H(I)$ of (2.13).

Lemma 2.1.5 *For a given $\xi \in [\xi_0, \xi_M]$, denote I^*_ξ as the optimal solution to the minimization problem $\min_{I \in \mathcal{I}_{p,0}, I(a)=\xi} H(I)$. Then, I^*_ξ can be summarized as follows:*

1. *If $\xi_0 \leq \xi \leq \xi_1$ and $\xi_0 < \xi_1$, then*

$$I^*_\xi(x) = x - (x-\xi)^+ + (x-a)^+ - (x-a-p)^+ + (x-d_{3,\xi})^+,$$

where $d_{3,\xi}$ is determined by $(1+\theta)\mathbb{E}\left[I^*_\xi(X)\right] = p$.

2. *If $\xi_1 \leq \xi \leq \xi_2$ and $\xi_1 < \xi_2$, then*

$$I^*_\xi(x) = (x-d_{1,\xi})^+ - (x-d_{1,\xi}-\xi)^+ + (x-a)^+ - (x-a-p)^+,$$

where $d_{1,\xi}$ is determined by $(1+\theta)\mathbb{E}\left[I^*_\xi(X)\right] = p$.

3. *If $\xi_2 \leq \xi \leq \xi_M$ and $\xi_2 < \xi_M$, then*

$$I^*_\xi(x) = (x-a+\xi)^+ - (x-a)^+ + (x-d_{2,\xi})^+ - (x-d_{2,\xi}-p)^+,$$

where $d_{2,\xi}$ is determined by $(1+\theta)\mathbb{E}\left[I^*_\xi(X)\right] = p$.

Remark 2.1.2 *It is possible that in some particular cases, one or two cases in Lemma 2.1.5 is invalid. However, it won't affect the completion of Lemma 2.1.5. For example, if $\xi_0 = \xi_1$, by definition it induces $\xi_0 = \xi_1 = \xi_2 = 0$ and $[\xi_0, \xi_M] = [0, \xi_M]$, where $\xi_M > 0$. Even when Cases 1 and 2 are invalid, the last case already covers the whole range $[\xi_0, \xi_M] = [\xi_2, \xi_M] = [0, \xi_M]$.*

For any $\xi \in [\xi_0, \xi_M]$, define

$$h(\xi) \triangleq H(I_\xi^*). \tag{2.20}$$

Lemma 2.1.5 implies that

$$\min_{I \in \mathcal{I}_{p,0}, I(a)=\xi} H(I) = H(I_\xi^*) = h(\xi).$$

The next theorem, which is the main result of this section, gives the optimal solution to Problem (2.3). Since $p = 0$ induces a trivial case that the only feasible reinsurance contract is zero contract, i.e. $I \equiv 0$, in order to avoid a redundant argument, we exclude this case in the next theorem.

Theorem 2.1.6 *Assume $0 < p < (1+\theta)\mathbb{E}[X]$. Then Problem (2.3) is equivalent to Problem (2.13) and the optimal solution to Problem 2.3, denoted by I^*, is summarized as follows:*

1. *If $\xi_1 = a$, the optimal solution is*

$$I^*(x) = x - (x - a - p)^+ + (x - d_{3,a})^+,$$

where $d_{3,a}$ is determined by $(1+\theta)\mathbb{E}[I^*(X)] = p$.

2. *If $\xi_1 < a$ and $h'(\xi_M) \leq 0$, then $\xi_M = a$ and the optimal solution is*

$$I^*(x) = x - (x-a)^+ + (x - d_{2,a})^+ - (x - d_{2,a} - p)^+,$$

where $d_{2,a}$ is determined by $(1+\theta)\mathbb{E}[I^*(X)] = p$.

3. *If $\xi_1 < a$ and $h'(\xi_M) > 0$ where $h(\cdot)$ is defined by (2.20), there exists $\xi^* \in [\xi_2, \xi_M]$ such that $h'(\xi^*) = 0$, and the optimal solution is*

$$I^*(x) = (x - a + \xi^*)^+ - (x-a)^+ + (x - d_{2,\xi^*})^+ - (x - d_{2,\xi^*} - p)^+,$$

where d_{2,ξ^*} is determined by $(1+\theta)\mathbb{E}[I^*(X)] = p$.

Remark 2.1.3 *All these three cases can be written into a unified formula*

$$I^*(x) = (x - d_1^*) - (x-a)^+ + (x - d_2^*)^+ - (x - d_2^* - p)^+ + (x - d_3^*)^+, \tag{2.21}$$

where

$$(d_1^*, d_2^*, d_3^*) = \begin{cases} (0, a, d_{3,a}), & \text{if} \quad \xi_1 = a; \\ (0, d_{2,a}, +\infty). & \text{if} \quad \xi_1 \le a \text{ and } h'(\xi_M) \le 0, \\ (a - \xi^*, d_{2,\xi^*}, +\infty), & \text{if} \quad \xi_1 < a \text{ and } h'(\xi_M) > 0. \end{cases}$$

Even though there are three variables d_1^*, d_2^* and d_3^* in the unified formula (2.21), in each particular case stated in Theorem 2.1.6, two of them will reduce to constants and the other one is determined by the premium condition.

Remark 2.1.4 *We point out that for a feasible contract $I \in \mathcal{I}_p$, if $I(x) \le \omega_I + P_I = I(a) + p$ for all $x \ge 0$, then the contract is a default risk-free contact, i.e., the insurer will not face default risk with this contract.*

In (1) of Theorem 2.1.6, which corresponds to the case where $\xi_1 = a$, if $p_{2,a} = p$, where $p_{2,\xi}$ is defined in (2.17), then $d_{3,a} = \infty$ and the optimal contract I^* is reduced to $I^*(x) = x - (x - a - p)^+ = I^*(x) \wedge (I^*(a) + p) \le I^*(a) + p$, namely the optimal contract is a default risk-free contract. However, if $p_{2,a} < p$, then there does not exist a default risk-free contract in $\mathcal{I}_p$. Indeed, suppose that $I \in \mathcal{I}_p$ is a default risk-free contract, then $I(x) \le I_{2,\xi}(x)$ for all $x \ge 0$, where $I_{2,\xi}$ is defined by (2.16) and $\xi = I(a)$. Since $\xi_1 = a$, by the definition of ξ_1 given in (2.18), we have $P_I \le p_{2,\xi} < p$. Thus, $I \notin \mathcal{I}_p$.

In (2) and (3) of Theorem 2.1.6, which correspond to the case where $\xi_1 < a$, it is obvious that the optimal solution I^* in both cases satisfies $I^*(x) \le I^*(a) + p$, namely the insurer will not face default risk with the two optimal contracts.

In summary, Theorem 2.1.6 suggests that, in order to lower default risk, an insurer should choose a contract without default risk as long as this kind of contract is available. This leads to limits for indemnities on the tails of the optimal contracts.

In addition, it has been mentioned that $I_0^* = (x - d^*)^+$ is the optimal solution to the classical Problem (2.5) in the absence of default risk. Note that $I_0^* \in \mathcal{I}_p$. It is easy to check that in all three cases of Theorem 2.1.6, the optimal contract I^* of Theorem 2.1.6 satisfies $\omega_{I^*} = I^*(a) > \omega_{I_0^*} = I_0^*(a) = (a - d^*)^+$, which means that the reinsurer will set up a higher initial reserve if the insurer chooses the optimal contract I^* of Theorem 2.1.6 than if the insurer chooses I_0^*. In this way, the insurer can reduce the default risk.

Remark 2.1.5 *In the classical model when no default risk is considered, the optimal solution is a stop-loss contract $I(x) = (x - d)^+$ where d is determined by the premium condition $(1 + \theta)\mathbb{E}[I(X)] = p$. However, Theorem 2.1.6 suggests that, in order to lower the default* *risk, the insurer should choose a contract which won't lead to the default, i.e. $I = \tilde{I}$, as long as this kind of contract is available (case 2 and case 3 in Theorem 2.1.6). This leads to a limit for the indemnity on the tail in the optimal contract.*

In case 2 and case 3 of Theorem 2.1.6, instead of one deductible at the beginning in the classical model, the optimal contract has two deductibles: one is at the beginning and the other one is in the middle. This could be viewed as a compromise between the default risk and the premium budget. By shifting a part of the deductible to the right, the optimal contract, which still satisfies the premium condition, requires the reinsurer to set up a higher initial reserve, which reduces the default risk for the insurer.

In case 1 of Theorem 2.1.6, since no contract satisfying $I = \tilde{I}$ is available based on the given premium p, the optimal contract should shift the entire deductible to the right of the default point $I(a) + p$ in order to have a full coverage before it.

2.2 Value-at-Risk Minimization

In this section, we consider the case when the insurer uses VaR at level β to measure its own risk. The optimal reinsurance problem becomes

$$\min_{I\in\mathcal{I}} \mathrm{VaR}_\beta \left(X - I(X) \wedge (\omega_I + P_I) + P_I\right) = \min_{I\in\mathcal{I}} V(I), \tag{2.22}$$

where $\omega_I \triangleq \mathrm{VaR}_\alpha(I(X))$, $P_I \triangleq (1+\theta)\mathbb{E}[I(X)]$ and

$$V(I) \triangleq \mathrm{VaR}_\beta \left(X - I(X) \wedge (\omega_I + P_I) + P_I\right).$$

It should be pointed out that the premium in Problem (2.22) is not fixed which is the case in Problem (2.3). Problem (2.22) reduces to a trivial problem if P_I is fixed.

For each $I \in \mathcal{I}$, the maximal payment $\omega_I + P_I$ is a constant. Denote $a \triangleq \mathrm{VaR}_\alpha(X)$, $b \triangleq \mathrm{VaR}_\beta(X)$ and

$$x_I \triangleq \sup\left\{x \leq 0 : I(x) < \omega_I + P_I\right\} \leq \infty.$$

If $x_I = \infty$, then $x - \tilde{I}(x) = x - I(x) = R(x)$ is a non-decreasing function on the non-negative real line. If $x_I < \infty$, then $I(x_I) = \omega_I + P_I$ and

$$x - \tilde{I}(x) = \begin{cases} R(x), & \text{for} \quad 0 \leq x \leq x_I; \\ x - (\omega_I + P_I), & \text{for} \quad x_I < x < \infty. \end{cases}$$

It is easy to see $x - \tilde{I}(x)$ is continuous on the non-negative real line and non-decreasing on $[0, x_I]$ and $[x_I, \infty)$ respectively. Thus $x - \tilde{I}(x)$ is non-decreasing on $[0, \infty)$. Indeed, for any $x \leq x_I < y$, we have

$$\begin{aligned} x - \tilde{I}(x) = R(x) &\leq R(x - I) = x_I - \tilde{I}(x_I) = x_I - (\omega_I + P_I) \\ &\leq y - (\omega_I + P_I) = y - \tilde{I}(y). \end{aligned}$$

Due to the translation invariance and preservation under continuous and non-decreasing function properties of VaR, the objective function $V(I)$ can be further simplified as follows:

$$V(I) = \mathrm{VaR}_\beta \left(X - I(X) \wedge (\omega_I + P_I)\right) + P_I = b - I(b) \wedge (I(a) + P_I) + P_I.$$

Similarly as in the utility function case, for each feasible reinsurance contract, we are going to construct a better one. The next lemma says that it is possible to find a modification with a particular form for each feasible reinsurance contract.

Lemma 2.2.1 *For any $I \in \mathcal{I}$, there exists reinsurance contract $m_I \in \mathcal{I}$ that satisfies $V(I) \leq V(m_I)$ and m_I has the form*

$$m_I(x) \;=\; (x - d_1)^+ - (x - a \wedge b)^+ + (x - d_2)^+ - (x - a \vee b)^+ \tag{2.23}$$

for some $(d_1, d_2) \in \mathbb{R}_+^2$ satisfying $0 \leq d_1 \leq a \wedge b \leq d_2 \leq a \vee b$.

Thanks to Lemma 2.2.1, the candidate set of optimal reinsurance contracts can be restricted to all contracts with the particular form given by expression (2.23). Denote

$$\mathcal{I}_{\alpha,\beta} \triangleq \{I \in \mathcal{I} : I \text{ has the expression (2.23) } \}.$$

Then $\mathcal{I}_{\alpha,\beta}$ is a finite-dimensional subset of $\mathcal{I}$. The optimal solution of the problem

$$\min_{I \in \mathcal{I}_{\alpha,\beta}} \mathrm{VaR}_\beta \left(X - I(X) \wedge (\omega_I + P_I) + P_I\right) \tag{2.24}$$

is also the optimal solution of Problem (2.22). To solve Problem (2.24), we rewrite it as a two-step minimization problem. There is a one-to-one mapping between $[0, a\wedge b] \times [a\wedge b, a\vee b]$ and $\mathcal{I}_{\alpha,\beta}$ through expression (2.23). For any pair $(d_1, d_2) \in [0, a \wedge b] \times [a \wedge b, a \vee b]$, define the function $v(d_1, d_2) \triangleq V(I)$ where $I \in \mathcal{I}_{\alpha,\beta}$ is associated with d_1 and d_2. Then

$$\min_{I \in \mathcal{I}_{\alpha,\beta}} V(I) = \min_{(d_1,d_2) \in [0,a\wedge b] \times [a\wedge b, a\vee b]} v(d_1, d_2).$$

For any fixed $d_1 \in [0, a \wedge b]$, if the function $v(d_1, \cdot)$ is continuous on the closed interval $[a \wedge b, a \vee b]$, then there exists $d_2^*(d_1) \in [a \wedge b, a \vee b]$ such that

$$\min_{d_2 \in [a \wedge b, a \vee b]} v(d_1, d_2) = v(d_1, d_2^*(d_1)). \tag{2.25}$$

If, moreover, $v(d_1, d_2^*(d_1))$ is continuous in d_1 on $[0, a \wedge b]$, then there exists $d_1^* \in [0, a \wedge b]$ such that

$$\min_{d_1 \in [0, a \wedge b]} v(d_1, d_2^*(d_1)) = v(d_1^*, d_2^*(d_1^*)). \tag{2.26}$$

For an arbitrary $(d_1, d_2) \in [0, a \wedge b] \times [a \wedge b, a \vee b]$,

$$v(d_1, d_2) \geq \min_{d_2 \in [a \wedge b, a \vee b]} v(d_1, d_2) = v(d_1, d_2^*(d_1)) \geq \min_{d_1 \in [0, a \wedge b]} v(d_1, d_2^*(d_1)) = v(d_1^*, d_2^*(d_1^*)).$$

Thus,

$$\min_{(d_1, d_2) \in [0, a \wedge b] \times [a \wedge b, a \vee b]} v(d_1, d_2) \geq v(d_1^*, d_2^*(d_1^*)).$$

On the other hand, $(d_1^*, d_2^*(d_1^*)) \in [0, a \wedge b] \times [a \wedge b, a \vee b]$ implies

$$\min_{(d_1, d_2) \in [0, a \wedge b] \times [a \wedge b, a \vee b]} v(d_1, d_2) \leq v(d_1^*, d_2^*(d_1^*)).$$

Therefore,

$$\min_{(d_1, d_2) \in [0, a \wedge b] \times [a \wedge b, a \vee b]} v(d_1, d_2) = v(d_1^*, d_2^*(d_1^*)) = \min_{d_1 \in [0, a \wedge b]} \left\{ \min_{d_2 \in [a \wedge b, a \vee b]} v(d_1, d_2) \right\},$$

which is a two-step minimization problem.

Theorem 2.2.2 *Denote I^* to be the optimal solution to Problem (2.22).*

1. *If $\alpha \leq \beta$, then*

$$I^*(x) \;=\; (x - b \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X))^+ - (x - b)^+ .$$

2. *If $\alpha \geq \beta$ and $\alpha \leq \frac{1}{1+\theta}$, there exists $d^* \in \mathbb{R}$ satisfying*

$$(1 + \theta) \int_{d^*}^{b} S_X(x) \mathrm{d}x = b - a. \tag{2.27}$$

Then

$$I^*(x) \;=\; (x - \max\{0, d^* \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\})^+ - (x - b)^+ .$$

Remark 2.2.1 *When $\alpha \leq \beta$, i.e. the reinsurer is more conservative than the insurer, the optimal contract is the same as the result in the classical model without default risk. Since the insurer only measures his total loss, based on VaR, at a lower level β, the initial reserve $\omega_I = \mathrm{VaR}_\alpha(X)$ set up by the reinsurer is high enough to ensure that the default risk has no impact to the optimal contract.*

On the contrary, if the insurer is more conservative ($\alpha > \beta$), in order to reduce the default risk, the insurer should require a lower deductible to force the reinsurer to set up a higher initial reserve. Moreover, from Equation 2.27, the larger the difference between the insurer and the reinsurer's risk tolerability, the smaller the deductible should be chosen by the insurer. For the case $\alpha > 1/(1+\theta)$, the optimal solution I^* has no closed form and the case is not interesting since in practice, α is a small value and usually $\alpha < 1/(1+\theta)$ holds.

2.3 Examples

In this section, numerical examples are provided for two optimization problems discussed in Section 2.1 and 2.2. Suppose X and Y are two random risks. Assume $X \sim Exp(\mu)$ and $Y \sim Pareto(\lambda, \gamma)$, i.e. $S_X(x) = e^{-x/\mu}$ for any $x \geq 0$ and $S_Y(y) = \left(\frac{\lambda}{y+\lambda}\right)^\gamma$ for any $y \geq 0$. All numerical results are given under the setting $\theta = 0.1$, $\mu = 100$, $\lambda = 200$ and $\gamma = 3$, i.e. X and Y has the same mean $\mu = \frac{\lambda}{\gamma-1} = 200$.

Example 2.3.1 *(Utility-based Maximization)*

Take the utility function $u(x) = x^2$. Consider exponential risk X with survival distribution $S_X(x) = e^{-x/\mu}$ for any $x \geq 0$. For a fixed premium value $p \in [0, (1+\theta)\mu]$, denote $\xi^*_{X,p}$ to be the zero root of the following function, if it exists:

$$
\begin{aligned}
h'(\xi) &= 2S_X(a-\xi)\left(d_{2,\xi} - a\right) - 2\left(\int_a^{d_{2,\xi}} + \int_{d_{2,\xi}+p}^\infty\right) S_X(x)\mathrm{d}x \\
&= 2\alpha e^{\xi/\mu}\left(d_{2,\xi} - a\right) - 2\mu\left(\alpha - e^{-d_{2,\xi}/\mu} + e^{-(d_{2,\xi}+p)/\mu}\right),
\end{aligned}
$$

where

$$
d_{2,\xi} = -\mu\left[\ln\left(\frac{p}{\mu(1+\theta)} - \alpha\left(e^{\xi/\mu} - 1\right)\right) - \ln\left(1 - e^{-p/\mu}\right)\right] \tag{2.28}
$$

According to Theorem 2.1.6, the optimal contract satisfies the following conditions.

1. Suppose p satisfies $\alpha e^{-p/\mu} + \frac{p}{\mu(1+\theta)} \geq 1$, then $\xi_1 = a$ and the optimal solution is

$$I^*(x) = x - (x - a - p)^+ + (x - d_{3,a})^+,$$

where

$$d_{3,a} = -\mu \ln \left(\frac{p}{\mu(1+\theta)} + \alpha e^{-p/\mu} - 1 \right) \geq a + p. \tag{2.29}$$

2. Suppose p satisfies $\alpha e^{-p/\mu} + \frac{p}{\mu(1+\theta)} < 1$, which implies the existence of $\xi^*_{X,p}$, and $\xi^*_{X,p} \geq a$, then $\xi_1 < a$ and $h'(\xi_M) \leq 0$. It implies the optimal solution is

$$I^*(x) = x - (x - a)^+ + (x - d_{2,a})^+ - (x - d_{2,a} - p)^+,$$

where $d_{2,a}$ is given by (2.28) when $\xi = a$.

3. Suppose p satisfies $\alpha e^{-p/\mu} + \frac{p}{\mu(1+\theta)} < 1$, which implies the existence of $\xi^*_{X,p}$, and $\xi^*_{X,p} < a$, then $\xi_1 < a$ and $h'(\xi_M) > 0$. It implies the optimal solution is

$$I^*(x) = (x - a + \xi^*_{X,p})^+ - (x - a)^+ + (x - d_{2,\xi^*_{X,p}})^+ - (x - d_{2,\xi^*_{X,p}} - p)^+,$$

where $d_{2,\xi^*_{X,p}}$ is given by (2.28) when $\xi = \xi^*_{X,p}$.

Using the unified formula given by (2.21), we get

$$I^*(x) \;\; = \;\; (x - d_1^*) - (x - a)^+ + (x - d_2^*)^+ - (x - d_2^* - p)^+ + (x - d_3^*)^+,$$

and the numerical results for (d_1^*, d_2^*, d_3^*) when $\alpha = 0.01$ (i.e. $\mathrm{VaR}_\alpha(X) = 460.517$) and $\alpha = 0.05$ (i.e. $\mathrm{VaR}_\alpha(X) = 299.573$) are summarized in Table 2.1 and Table 2.2 respectively.

Consider a Pareto risk Y with survival distribution $S_Y(y) = \left(\frac{\lambda}{y+\lambda}\right)^\gamma$, for any $x \geq 0$. For a fixed premium value $p \in [0, (1+\theta)\frac{\lambda}{\gamma-1}]$, denote $\xi^*_{Y,p}$ be the zero root of the following function, if it exists,

$$\begin{aligned} h'(\xi) &= 2S_Y(a - \xi)\left(d_{2,\xi} - a\right) - 2\left(\int_a^{d_{2,\xi}} + \int_{d_{2,\xi}+p}^{\infty}\right) S_Y(y)\mathrm{d}y \\ &= 2\left(d_{2,\xi} - a\right)\left(\frac{\lambda}{a - \xi + \lambda}\right)^\gamma - \frac{2\lambda}{\gamma - 1}\left(\alpha^{\frac{\gamma-1}{\gamma}} - \left(\frac{\lambda}{d_{2,\xi} + \lambda}\right)^{\frac{\gamma-1}{\gamma}} + \left(\frac{\lambda}{d_{2,\xi} + \lambda + p}\right)^{\frac{\gamma-1}{\gamma}}\right), \end{aligned}$$

Table 2.1: Exponential Risk X & $\alpha = 0.01$

p	d_1^*	d_2^*	d_3^*
80	31.225	461.168	∞
99.2	9.921	460.940	∞
105.88	3.456	460.806	∞
108.1	1.396	460.809	∞
109.631	0	460.811	∞
109.8	0	460.517	649.089

Table 2.2: Exponential Risk X & $\alpha = 0.05$

p	d_1^*	d_2^*	d_3^*
80	28.691	302.681	∞
99.2	8.229	301.653	∞
105.88	1.971	301.407	∞
108.1	0	301.332	∞
109.631	0	299.573	431.620
109.8	0	299.573	420.917

where $d_{2,\xi}$ satisfies

$$p = (1+\theta)\left(\int_{a-\xi}^{a} + \int_{d_{2,\xi}}^{d_{2,\xi}+p}\right)\left(\frac{\lambda}{y+\lambda}\right)^{\gamma} \mathrm{d}y. \tag{2.30}$$

According to Theorem 2.1.6, the optimal contract has the form

1. Suppose p satisfies $1 - \left(\frac{\lambda}{a+p+\lambda}\right)^{\gamma-1} \leq \frac{p(\gamma-1)}{\lambda(1+\theta)}$, then $\xi_1 = a$. It implies the optimal solution is
$$I^*(x) = x - (x-a-p)^+ + (x-d_{3,a})^+,$$
where
$$d_{3,a} = -\lambda + \left[\frac{p(\gamma-1)}{\lambda^{\gamma}(1+\theta)} - \lambda^{-\gamma+1} + (a+p+\lambda)^{-\gamma+1}\right]^{-\frac{1}{\gamma-1}} \geq a+p.$$

2. Suppose p satisfies $1 - \left(\frac{\lambda}{a+p+\lambda}\right)^{\gamma-1} > \frac{p(\gamma-1)}{\lambda(1+\theta)}$, which implies the existence of $\xi^*_{Y,p}$, and if $\xi^*_{Y,p} \geq a$, then $\xi_1 < a$ and $h'(\xi_M) \geq 0$. It implies the optimal solution is
$$I^*(x) = x - (x-a)^+ + (x-d_{2,a})^+ - (x-d_{2,a}-p)^+,$$

where $d_{2,a}$ is given by (2.30) when $\xi = a$.

3. Suppose p satisfies $1 - \left(\frac{\lambda}{a+p+\lambda}\right)^{\gamma-1} > \frac{p(\gamma-1)}{\lambda(1+\theta)}$, which implies the existence of $\xi^*_{Y,p}$, and $\xi^*_{Y,p} < a$, then $\xi_1 < a$ and $h'(\xi_M) > 0$. It implies the optimal solution is

$$I^*(x) = (x - a + \xi^*_{Y,p})^+ - (x-a)^+ + (x - d_{2,\xi^*_{Y,p}})^+ - (x - d_{2,\xi^*_{Y,p}} - p)^+,$$

where $d_{2,\xi^*_{Y,p}}$ is given by (2.30) when $\xi = \xi^*_{Y,p}$.

We summarize all these three cases in the unified formula (2.21) and obtain

$$I^*(x) = (x - d_1^*) - (x-a)^+ + (x - d_2^*)^+ - (x - d_2^* - p)^+ + (x - d_3^*)^+.$$

The numerical results for (d_1^*, d_2^*, d_3^*) when $\alpha = 0.01$ (i.e. $\mathrm{VaR}_\alpha(Y) = 728.318$) and $\alpha = 0.05$ (i.e. $\mathrm{VaR}_\alpha(Y) = 342.884$) are summarized in Table 2.3 and Table 2.4 respectively.

Table 2.3: Pareto Risk Y & $\alpha = 0.01$

p	d_1^*	d_2^*	d_3^*
80	28.405	734.196	∞
99.2	6.305	732.488	∞
105.88	0	732.1067	∞
108.1	0	728.318	1215.4
109.631	0	728.3	888.275
109.8	0	728.3	864.518

Table 2.4: Pareto Risk Y & $\alpha = 0.05$

p	d_1^*	d_2^*	d_3^*
80	19.2	356.748	∞
99.2	0	352.764	∞
105.88	0	342.9	633.469
108.1	0	342.884	520.208
109.631	0	342.884	464.486
109.8	0	342.884	459.096

Remark 2.3.1 *Comparisons between Table 2.1 and Table 2.2 or between Table 2.3 and Table 2.4 suggest that when the risk level is fixed, a higher premium leads to a larger*

optimal contract. When the premium is fixed, if the risk level is lower (or a larger α), the maximal payment ability of the reinsurer is weaker and it leads to a smaller d_1, i.e. the optimal contract should have a smaller deductible in order to have a larger coverage before the default.

Comparison between Table 2.1 and Table 2.3 or between Table 2.2 and Table 2.4 suggest that when both premium and risk level α are same, Pareto random loss Y who has a relative heavy tail has a larger coverage than Exponential random loss X.

Remark 2.3.2 (Extreme Cases) *We use exponential risk X as an illustration. The same argument can be done for Y. The expression (2.29) implies, as $p \uparrow (1+\rho)\mu$, we have $d_{3,a} \downarrow a+p = 570.517$ and thus in the extreme case when $p = (1+\rho)\mu$ the optimal contract is $I^*(x) = x$, for all $x \geq 0$, i.e. I^* is the full reinsurance. This is consistent with the fact that $I^*(x) = x$ is the only feasible contract when $p = (1+\rho)\mu$. In another extreme case, as $p \downarrow 0$, we have $\alpha e^{-p/\mu} + \frac{p}{\mu(1+\theta)} \downarrow 0$ and $\xi_M \downarrow 0$. Thus, for small enough premium value p, the optimal contract is*

$$I^*(x) = (x - a + \xi^*_{X,p})^+ - (x-a)^+ + (x - d_{2,\xi^*_{X,p}})^+ - (x - d_{2,\xi^*_{X,p}} - p)^+,$$

where $d_{2,\xi^*_{X,p}}$ is given by (2.28) when $\xi = \xi^*_{X,p}$. Thus, $I^*(x) \downarrow 0$ for all $x \geq 0$ as $p \downarrow 0$. It is consistent with the fact that the only feasible contract is the zero reinsurance contract when the premium budget is zero.

Example 2.3.2 *(VaR-based Minimization)*

Denote I^*_X and I^*_Y as the optimal reinsurance contracts for X and Y respectively. According to Theorem 2.2.2,

$$I^*_X(x) = (x - d_X)^+ - (x - \mathrm{VaR}_\beta(X))^+ \text{ and } I^*_Y(y) = (y - d_Y)^+ - (y - \mathrm{VaR}_\beta(Y))^+,$$

where

1. if $\alpha \leq \beta$,

$$d_X = \mathrm{VaR}_{\frac{1}{1+\theta}}(X) \wedge \mathrm{VaR}_\beta(X) \text{ and } d_Y = \mathrm{VaR}_{\frac{1}{1+\theta}}(Y) \wedge \mathrm{VaR}_\beta(Y);$$

2. if $\beta \leq \alpha \leq \frac{1}{1+\theta}$,

$$d_X = \max\left\{0, d^*_X \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\right\} \text{ and } d_Y = \max\left\{0, d^*_Y \wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(Y)\right\},$$

and d_X^* and d_Y^* solve the following equations respectively

$$\begin{aligned}
-\mu \ln \beta - (-\mu \ln \alpha) &= (1+\theta) \int_{d_X^*}^{-\mu \ln \beta} e^{-\mu x} \mathrm{d}x, \\
\lambda\left(\beta^{-1/\gamma} - 1\right) - \lambda\left(\alpha^{-1/\gamma} - 1\right) &= (1+\theta) \int_{d_Y^*}^{\lambda\left(\beta^{-1/\gamma}-1\right)} \left(\frac{\lambda}{y+\lambda}\right)^{\gamma} \mathrm{d}y.
\end{aligned}$$

Table 2.5 gives the numerical results for d_X and d_Y.

Table 2.5: Deductible Values

(α, β)	d_X	$\mathrm{VaR}_\beta(X)$	d_Y	$\mathrm{VaR}_\beta(Y)$
(0.0100, 0.050)	9.5310	299.5732	6.4560	342.8835
(0.0100, 0.028)	9.5310	357.5551	6.4560	458.6338
(0.0185, 0.015)	9.5310	419.9705	6.4560	610.9603
(0.050, 0.0100)	0	460.5170	0	728.3178
(0.028, 0.0100)	5.5494	460.5170	0	728.3178
(0.028, 0.0185)	9.5310	398.9985	4.4483	556.2048
(0.015, 0.0185)	9.5310	398.9985	6.4560	556.2048

Remark 2.3.3 *From Table 2.5, random loss Y with Pareto distribution, who has a heavy tail, always has a lower deductible than the random loss X with exponential distribution, who has a light tail. This comparison is consistent with the classical result when no default risk is involved.*

2.4 Appendix

Proof of Lemma 2.1.2. Let I be an arbitrary reinsurance contract from $\mathcal{I}_p$. Denote $\xi \triangleq I(a)$ and

$$x_I \triangleq \sup\{x \geq 0 : I(x) < \xi + p\} \in (a, \infty].$$

Since I is non-decreasing and continuous on $[0, \infty)$, one has $\omega_I = \mathrm{VaR}_\alpha(I(X)) = I(\mathrm{VaR}_\alpha(X)) = \xi$. Therefore, after the consideration of reinsurer's default risk, the insurer has the real reinsurance contract

$$\tilde{I}(x) = I(x) \wedge (\xi + p), \text{ for } x \geq 0.$$

Note that

$$\begin{aligned} H(I) =& \mathbb{E}\left[u\left(X - \tilde{I}(X)\right)\right] \\ =& \mathbb{E}\left[u\left(X - I(X)\right)|0 \leq X < a\right]\mathbb{P}(0 \leq X < a) \\ &+ \mathbb{E}\left[u\left(X - I(X)\right)|a \leq X < x_I\right]\mathbb{P}(a \leq X < x_I) \\ &+ \mathbb{E}\left[u\left(X - \xi - p\right)\right]\mathbb{P}(X \geq x_I). \end{aligned}$$

Firstly, we are going to construct k_I on interval $[0, a)$, $[a, x_I)$ and $[x_I, \infty)$ separately.

1) For $0 \leq x < a$, define $k_d(x) \triangleq \max\{(x - d)^+, \xi\}$, with respect to each $d \in [0, a - \xi]$. When $d = 0$, we have $k_0(x) = x \vee \xi$ and thus $k_0(x) \geq I(x)$ on the interval $[0, a)$ and $\mathbb{E}[k_0(X)|0 \leq X < a] \geq \mathbb{E}[I(X)|0 \leq X < a]$. When $d = a - \xi$, we have $k_{a-\xi}(x) = (x - a + \xi)^+$ and thus $k_{a-\xi}(x) \leq I(x)$ on $[0, a)$ and $\mathbb{E}[k_{a-\xi}(X)|0 \leq X < a] \leq \mathbb{E}[I(X)|0 \leq X < a]$. It is obvious that $\mathbb{E}[k_d(X)|0 \leq X < a]$ is continuous in d. Therefore, there exists $d_1 \in [0, a - \xi]$ such that $\mathbb{E}[k_{d_1}(X)|0 \leq X < a] = \mathbb{E}[I(X)|0 \leq X < a]$. Define $k_I(x) = k_{d_1}(x)$ on the interval $[0, a]$.

2) For $a \leq x < x_I$, define $k_I(x) \triangleq \max\{\xi + (x - d_2)^+, \xi + p\}$, where $d_2 \in [a, x_I]$ is such that $\mathbb{E}[I(X)|a \leq X < x_I] = \mathbb{E}[k_I(X)|a \leq X < x_I]$. The existence of d_2 can be shown by the similar argument in 1).

3) When $x_I < \infty$, for $x \geq x_I$, define $k_I(x) \triangleq \xi + p + (x - d_3)^+$, where $d_3 \geq x_I$ is such that $\mathbb{E}[I(X)|X \geq x_I] = \mathbb{E}[k_I(X)|X \geq x_I]$ and the existence of d_3 can be shown by the similar argument in 1). When $x_I = \infty$, intervals $[0, a)$ and $[a, x_I)$ already cover the whole non-negative real line.

It is easy to see by the construction of k_I that

$$\begin{aligned}\mathbb{E}\left[k_I(X)\right] &= \mathbb{E}\left[u\left(k_I(X)\right)|0 \le X < a\right]\mathbb{P}(0 \le X < a) \\ &\quad +\mathbb{E}\left[u\left(k_I(X)\right)|a \le X < x_I\right]\mathbb{P}(a \le X < x_I) \\ &\quad +\mathbb{E}\left[u\left(k_I(x)\right)\right]\mathbb{P}(X \ge x_I) \\ &= \mathbb{E}\left[I(X)\right] = p,\end{aligned}$$

and thus $k_I \in \mathcal{I}_p$. It remains to show that $H(k_I) \le H(I)$. Define a random variable X_1 with distribution function $F_1(x) = \frac{F_X(x)}{F_X(a)}$, for $0 \le x < a$. Then for any Borel measurable function $b(\cdot)$, one has

$$\mathbb{E}\left[b(X)|0 \le X < a\right] = \int_0^a \frac{b(x)}{F_X(a)}\mathrm{d}F_X(x) = \int_0^a b(x)\mathrm{d}F_1(x) = \mathbb{E}\left[b(X_1)\right].$$

Note that, on the interval $[0, a)$, functions $I(x)$ can cross $k_I(x)$ from above at most once, say $c_1 \in [0, a)$. Thus,

$$\begin{aligned} x - k_I(x) &\ge x - I(x) \quad \text{for} \quad x < c_1; \\ \text{and } x - k_I(x) &\le x - I(x) \quad \text{for} \quad x \ge c_1. \end{aligned}$$

Together with $\mathbb{E}\left[I(X_1)\right] = \mathbb{E}\left[k_I(X_1)\right]$ and $x - k_I(x)$ and $x - I(x)$ are both continuous and non-decreasing functions, Lemma 2.1.1 implies $X_1 - k_I(X_1) \le_{cx} X_1 - I(X_1)$. Therefore, for any convex function u,

$$\begin{aligned}\mathbb{E}\left[u\left(X - I(X)\right)|0 \le X < a\right] &= \mathbb{E}\left[u\left(X_1 - I(X_1)\right)\right] \\ &\ge \mathbb{E}\left[u\left(X_1 - k_I\left(X_1\right)\right)\right] = \mathbb{E}\left[u\left(X - k_I\left(X\right)\right)|0 \le X < a\right].\end{aligned}$$

By using the same argument, it is easy to see that

$$\begin{aligned}\mathbb{E}\left[u\left(X - I(X)\right)|a \le X < x_I\right] &\ge \mathbb{E}\left[u\left(X - k_I(X)\right)|a \le X < x_I\right], \text{ and} \\ \mathbb{E}\left[u\left(X - I(X)\right)|X \ge x_I\right] &\ge \mathbb{E}\left[u\left(X - k_I(X)\right)|X \ge x_I\right].\end{aligned}$$

Therefore, on the whole support of random loss X, we can conclude that $H(k_I) \le H(I)$. ■

Proof of Lemma 2.1.3.

"(1) ⇔ (2) " Suppose $\{I \in \mathcal{I}_{p,0} : I(a) = \xi\} \ne \emptyset$. For any $I \in \{I \in \mathcal{I}_{p,0} : I(a) = \xi\}$, it is easy to see that the following inequalities hold: $I_{0,\xi}(x) \le I(x) \le I_{M,\xi}(x)$ for all $x \ge 0$. It implies that $P_{I_{0,\xi}} \le P_I \le P_{I_{M,\xi}}$, that is $p_{0,\xi} \le p \le p_{M,\xi}$. Conversely, suppose $\xi \in [0, a]$

satisfies the inequality $p_{0,\xi} \le p \le p_{M,\xi}$. The premium of any $I \in \mathcal{I}$ satisfies expression (2.4) and $I(a) = \xi$ can be written as a continuous function of (d_1, d_2, d_3) as follows:

$$P_I = P(d_1, d_2, d_3) = (1+\theta)\mathbb{E}[I(X)] = (1+\theta)\left(\int_{d_1}^{d_1+\xi} + \int_{d_2}^{d_2+p} + \int_{d_3}^{\infty}\right) S_X(x)\mathrm{d}x,$$

where $0 \le d_1 \le d_1 + I(a) \le a \le d_2 \le d_2 + p \le d_3 \le \infty$. Note that when $(d_1, d_2, d_3) = (a-\xi, \infty, \infty)$, we have $P(a-\xi, \infty, \infty) = P_{I_{0,\xi}} = p_{0,\xi}$, and when $(d_1, d_2, d_3) = (0, a, a+p)$, we have $P(0, a, a+p) = P_{I_{M,\xi}} = p_{M,\xi}$. Since the function $P(d_1, d_2, d_3)$ is continuous in (d_1, d_2, d_3), it goes through all values of the interval $[p_{0,\xi}, p_{M,\xi}]$ and thus there exists $(d_{1,\xi}, d_{2,\xi}, d_{3,\xi})$ such that $P(d_{1,\xi}, d_{2,\xi}, d_{3,\xi}) = p$. Therefore, the contract I which has the expression (2.4) with $(d_1, d_2, d_3, I(a)) = (d_{1,\xi}, d_{2,\xi}, d_{3,\xi}, \xi)$ satisfies $P_I = p$, and it implies that $I \in \{I \in \mathcal{I}_{p,0} : I(a) = \xi\} \neq \emptyset$.

"(2) $\Leftrightarrow$ (3) " The equivalence of (2) and (3) is given by the definitions (2.11) and (2.12).

Any two contracts I_1 and I_2 in $\mathcal{I}_{p,0}$ are viewed as the same if they are equal almost everywhere with repsect to Lebesgue's measure. Suppose $I_1(a) \neq I_2(a)$, since both contracts are continuous at point a, there exists $\delta > 0$ such that $I_1(x) \neq I_2(x)$ on the open interval $(a-\delta, a+\delta)$ and thus I_1 and I_2 are not the same. As a consequence, for $\xi_1 \neq \xi_2$,

$$\{I \in \mathcal{I}_{p,0} : I(a) = \xi_1\} \bigcap \{I \in \mathcal{I}_{p,0} : I(a) = \xi_2\} = \emptyset.$$

It has been proved that $\{I \in \mathcal{I}_{p,0} : I(a) = \xi\} = \emptyset$ for any $\xi \in [0, a] \setminus [\xi_0, \xi_M]$, thus,

$$\mathcal{I}_{p,0} = \bigcup_{0 \le \xi \le a} \{I \in \mathcal{I}_{p,0} : I(a) = \xi\} = \bigcup_{\xi_0 \le \xi \le \xi_M} \{I \in \mathcal{I}_{p,0} : I(a) = \xi\},$$

namely, $\mathcal{I}_{p,0}$ can be written as the union of disjoint non-empty sets. ■

Proof of Lemma 2.1.4. It is easy to see that for any $\xi \in [\xi_0, \xi_M]$, we have $I_{0,\xi}(x) \le I_{1,\xi}(x) \le I_{2,\xi}(x) \le I_{M,\xi}(x)$ for all $x \ge 0$. Therefore, $p_{0,\xi} \le p_{1,\xi} \le p_{2,\xi} \le p_{M,\xi}$. Note that, p is assumed to satisfies $0 < p < (1+\theta)\mathbb{E}[X]$. Since all proofs are similar, we only prove the case (1) and the other two cases are omitted.

(1) Suppose $\xi_0 = \xi_1$. First, we are going to show that $\xi_0 = 0$. Suppose not, namely $\xi_0 > 0$. By the definition (2.11) of ξ_0, we have $p_{M,0} < p$. Note that $p_{M,a} = (1+\theta)\mathbb{E}[X] \ge p$ by our assumption. Obviously, $p_{M,\xi}$ is continuous and increasing in $\xi \in [0, a]$, thus, $p_{M,\xi_0} = p$. By (2.17) and (2.10), we have that $p_{M,\xi} - p_{2,\xi} = (1+\theta)\int_{a+p}^{\infty} S_X(x)\mathrm{d}x > 0$ for any $\xi \in [0, a]$, and in particular, $p_{2,\xi_0} < p_{M,\xi_0} = p$. It is easy to see that $p_{2,\xi}$ is continuous and increasing in $\xi \in [\xi_0, \xi_M]$. Thus, there exists $\delta > 0$ such that $p_{2,\xi} < p$ for any $\xi \in (\xi_0, \xi_0 + \delta)$. According

to the definition (2.18), we get that $\xi_1 > \xi_0$ which contradicts with the assumption that $\xi_0 = \xi_1$. Therefore, $\xi_0 = 0 = \xi_1$ and moreover $p_{M,0} > p_{2,0} \geq p$. Second, we show that $\xi_2 = 0$. From (2.15) and (2.17), we have $p_{1,0} = (1+\theta)\int_a^{a+p} S_X(x)\mathrm{d}x = p_{2,0} \geq p$. That is, for any $\xi > 0$, we have $p_{1,\xi} \geq p_{1,0} \geq 0$, and then, $\xi_2 = 0$ by its definition (2.19). Finally, we need to show that $0 < \xi_M$. Indeed, $p_{0,0} = 0 < p$. Since $p_{0,\xi}$ is continuous and increasing in $\xi \in [0,a]$, there exists $\delta_1 > 0$ such that $p_{0,\xi} < p$ for any $\xi \in [0, \delta_1)$. Therefore, by definition (2.12), we get $0 < \xi_M$. ■

Proof of Lemma 2.1.5. For any $I \in \mathcal{I}_{p,0}$, i.e. I has expression (2.4) for some (d_1, d_2, d_3, ξ), the corresponding realized indemnity is

$$\tilde{I}(x) \triangleq I(x) \wedge (w_I + P_I) = (x-d_1)^+ - (x-(d_1+\xi))^+ + (x-d_2)^+ - (x-(d_2+p))^+.$$

Note that $\tilde{I}(x) = I(x)$ for $0 \leq x \leq d_2 + p$. Thus, the objective function is

$$\begin{aligned} H(I) = \mathbb{E}\left[u\left(X - \tilde{I}(X)\right)\right] &= \int_0^\infty u(x - \tilde{I}(x))\mathrm{d}F_X(x) \qquad (2.31) \\ &= \int_0^\infty S_X(x)u'(x-\tilde{I}(x))\left(1 - \tilde{I}'(x)\right)\mathrm{d}x \\ &= u(0) + \left(\int_0^{d_1} + \int_{d_1+\xi}^{d_2}\right) S_X(x)u'\left(x - I(x)\right)\mathrm{d}x \\ &\quad + \int_{d_2+p}^\infty S_X(x)u'\left(x-\xi-p\right)\mathrm{d}x. \end{aligned}$$

where $u(0)$ is a constant and $\mathbb{E}\left[u\left(X-\tilde{I}(X)\right)\right]$ is assumed to be exist and thus the integration on the right hand side of the second equality is finite. Note that, for any $I \in \mathcal{I}_{p,0}$, the value $H(I)$ does not depend on d_3. Indeed, the coefficient d_3 is only used to adjust the expectation of I to match the expectation condition

$$(1+\theta)\mathbb{E}\left[I(X)\right] = (1+\theta)\left(\int_{d_1}^{d_1+\xi} + \int_{d_2}^{d_2+p} + \int_{d_3}^{\infty}\right) S_X(x)\mathrm{d}x = p. \qquad (2.32)$$

Now, we are going to optimize $H(I)$ using the expression (2.31) and the restriction (2.32) and to find the minimizers of d_i, denoted by $d_{i,\xi}$, $i = 1,2,3$.

(1) Suppose $\xi_0 \leq \xi \leq \xi_1$ (if $\xi_0 < \xi_1$) or equivalently $p_{2,\xi} \leq p \leq p_{M,\xi}$. There exists $a \leq d_{3,\xi} \leq \infty$ such that

$$I_\xi^*(x) = x - (x-a+\xi)^+ + (x-a)^+ - (x-a-p)^+ + (x-d_{3,\xi})^+$$

satisfies the premium condition $(1+\theta)\mathbb{E}[I_\xi^*(X)] = p$. Thus, I_ξ^* has form (2.4) with $(d_1, d_2, d_3) = (0, a, d_{3,\xi})$. Moreover, I_ξ^* is the optimal solution because for any $I \in \mathcal{I}_{p,0}$ satisfying $I(a) = \xi$, one has $\tilde{I}_\xi^*(x) \geq \tilde{I}(x)$ for all $x \geq 0$.

Suppose $\xi_1 \leq \xi \leq \xi_M$ (if $\xi_1 < \xi_M$) or equivalently $p_{0,\xi} \leq p \leq p_{2,\xi}$. In this case, there exist d_1 and d_2 such that

$$I(x) = (x-d_1)^+ - (x-d_1-\xi)^+ + (x-d_2)^+ - (x-d_2-p)^+ \tag{2.33}$$

satisfies the premium condition $(1+\theta)\mathbb{E}\left[I(X)\right] = p$. Note that expression (2.4) is reduced to expression (2.33) when $d_3 = \infty$. We claim that the minimizer I_ξ^* should have the form (2.33). This claim can be proved using the following arguments.

For any $I \in \mathcal{I}_{p,0}$ with $d_3 < \infty$, if there exists $\tilde{d}_2 \geq a$ such that $I_1(x) = (x-d_1)^+ - (x-d_1-\xi)^+ + (x-\tilde{d}_2)^+ - (x-\tilde{d}_2-p)^+ \in \mathcal{I}_{p,0}$, then the following equation

$$0 = \mathbb{E}[I_1(X)] - \mathbb{E}[I(X)] = \int_{d_2}^{d_2+p} \left[S_X(x+\tilde{d}_2-d_2) - S_X(x)\right] \mathrm{d}x - \int_{d_3}^{\infty} S_X(x)\mathrm{d}x,$$

implies that $\tilde{d}_2 < d_2$ and thus $\tilde{I}_1(x) - \tilde{I}(x) = I_1(x) \wedge (I_1(a)+p) - I(x) \wedge (I(a)+p) \geq 0$ for all $x \geq 0$. If such $\tilde{d}_2$ does not exist, then there exists $0 \leq \tilde{d}_1 \leq d_1$ such that $I_1(x) = (x-\tilde{d}_1)^+ - (x-\tilde{d}_1-\xi)^+ + (x-a)^+ - (x-a-p)^+ \in \mathcal{I}_{p,0}$ and thus $\tilde{I}_1(x) \geq \tilde{I}(x)$ for all $x \geq 0$. In short, we can find another contract I_1 of the form (2.33) in $\mathcal{I}_{p,0}$ such that $\tilde{I}_1(x) \geq \tilde{I}(x)$ for all $x \geq 0$ and thus $H(I_1) \leq H(I)$. Therefore, the insurer should choose the reinsurance contract satisfying the form (2.33) or $I = \tilde{I}$.

For any contract $I \in \mathcal{I}$ of the form (2.33), the premium condition

$$\left(\int_{d_1}^{d_1+\xi} + \int_{d_2}^{d_2+p}\right) S_X(x)\mathrm{d}x = \frac{p}{1+\theta}$$

implies that d_2 can be written as an implicit function of d_1, i.e. $d_2 = d_2(d_1)$. It is not hard to see $d_2(d_1)$ is a non-increasing function of d_1 and its derivative satisfies

$$S_X(d_1+\xi) - S_X(d_1) + \left(S_X(d_2+p) - S_X(d_2)\right) d_2'(d_1) = 0.$$

The objective function given by expression (2.31) now depends on d_1 only and thus denote it as a one variable function $H_\xi(d_1)$. Taking the derivative with respect to d_1, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}d_1} H_\xi(d_1) &= u'(d_1)\left(S_X(d_1) - S_X(d_1+\xi)\right) + u'(d_2-\xi)\left(S_X(d_2) - S_X(d_2+P)\right) d_2'(d_1) \\
&= \left(u'(d_1) - u'(d_2-\xi)\right)\left(S_X(d_1) - S_X(d_1+\xi)\right) \\
&\leq 0.
\end{aligned}$$

Therefore, one should choose d_1 as large as possible to have the smallest expectation of the utility.

(2.1) When $\xi_1 \leq \xi \leq \xi_2$ (if $\xi_1 < \xi_2$) or equivalently $p_{1,\xi} \leq p \leq p_{2,\xi}$, the largest possible value for d_1 is $d_{1,\xi}$ which satisfies $d_2(d_{1,\xi}) = a$ and the corresponding optimal solution is

$$I_\xi^*(x) \;=\; (x - d_{1,\xi})^+ - (x - d_{1,\xi} - \xi)^+ + (x - a)^+ - (x - a - p)^+.$$

Then I_ξ^* is of the form (2.4) with $(d_1, d_2, d_3) = (d_{1,\xi}, a, \infty)$ where $d_{1,\xi}$ is determined by the expectation condition $(1+\theta)\mathbb{E}\left[I_\xi^*(X)\right] = p$.

(2.1) When $\xi_2 \leq \xi \leq \xi_M$ (if $\xi_2 < \xi_M$) or equivalently $p_{0,\xi} \leq p \leq p_{1,\xi}$, the largest possible value for d_1 is $a - \xi$ and the corresponding optimal solution is

$$I_\xi^*(x) \;=\; (x - a + \xi)^+ - (x - a)^+ + (x - d_{2,\xi})^+ - (x - d_{2,\xi} - p)^+,$$

where $d_{2,\xi} = d_2(a - \xi)$, namely I_ξ^* has form (2.4) with $(d_1, d_2, d_3) = (a - \xi,\, d_{2,\xi},\, \infty)$. ■

Proof of Theorem 2.1.6. Define function $h(\xi)$ on $[\xi_0, \xi_M]$ as follows:

$$h(\xi) \triangleq \min_{I \in \mathcal{I}_p,\, I(a)=\xi} \mathbb{E}[u(X - I(X) \wedge (\xi + p))] = \mathbb{E}[u(X - I_\xi^*(X) \wedge (\xi + p))].$$

From the results of Lemma 2.1.5, we discuss the following cases.

1. When $\xi \in [\xi_0, \xi_1]$ (if $\xi_0 < \xi_1$), one has $d_{3,\xi}$ is an increasing function of ξ and

$$h(\xi) = \int_\xi^a S_X(x) u'(x - \xi) \mathrm{d}x + \int_{a+p}^\infty S_X(x) u'(x - \xi - p) \mathrm{d}x.$$

   Clearly,

$$h'(\xi) = -\int_\xi^a S_X(x) u''(x - \xi) \mathrm{d}x - S_X(\xi) u'(0) - \int_{a+p}^\infty S_X(x) u''(x - \xi - p) \mathrm{d}x \leq 0,$$

   because $u(\cdot)$ is an increasing convex function with $u'(x) \geq 0$ and $u''(x) \geq 0$.

2. When $\xi \in [\xi_1, \xi_2]$ (if $\xi_1 < \xi_2$), we have

$$h(\xi) = \int_0^{d_{1,\xi}} S_X(x) u'(x) \mathrm{d}x + \int_{d_{1,\xi}+\xi}^a S_X(x) u'(x - \xi) \mathrm{d}x + \int_{a+p}^\infty S_X(x) u'(x - \xi - p) \mathrm{d}x.$$

The premium condition implies that $d_{1,\xi}$ can be written as an implicit function of ξ using the equation

$$(1+\theta)\left(\int_{d_{1,\xi}}^{d_{1,\xi}+\xi} + \int_{a}^{a+p}\right) S_X(x)\mathrm{d}x = p.$$

By taking the derivative with respect to ξ on both sides of the equation, we get

$$\frac{\mathrm{d}}{\mathrm{d}\xi}d_{1,\xi} = \frac{S_X\left(d_{1,\xi}+\xi\right)}{S_X(d_{1,\xi}) - S_X(d_{1,\xi}+\xi)} \geq 0.$$

It leads to

$$\begin{aligned} h'(\xi) &= S_X(d_{1,\xi})u'(d_{1,\xi})\frac{\mathrm{d}}{\mathrm{d}\xi}d_{1,\xi} - S_X(d_{1,\xi}+\xi)u'(d_{1,\xi})\left(1+\frac{\mathrm{d}}{\mathrm{d}\xi}d_{1,\xi}\right) \\ &\quad - \int_{d_{1,\xi}+\xi}^{a} S_X(x)u''(x-\xi)\mathrm{d}x - \int_{a+p}^{\infty} S_X(x)u''(x-\xi-p)\mathrm{d}x \\ &= -\int_{d_{1,\xi}+\xi}^{a} S_X(x)u''(x-\xi)\mathrm{d}x - \int_{a+p}^{\infty} S_X(x)u''(x-\xi-p)\mathrm{d}x \\ &\leq 0. \end{aligned}$$

3. When $\xi \in [\xi_2, \xi_M]$ (if $\xi_2 < \xi_M$),, we have

$$h(\xi) = \int_0^{a-\xi} S_X(x)u'(x)\mathrm{d}x + \int_a^{d_{2,\xi}} S_X(x)u'(x-\xi)\mathrm{d}x + \int_{d_{2,\xi}+p}^{\infty} S_X(x)u'(x-\xi-p)\mathrm{d}x.$$

Premium condition implies that $d_{2,\xi}$ can be written as an implicit function of ξ by the equation

$$(1+\theta)\left(\int_{a-\xi}^{a} + \int_{d_{2,\xi}}^{d_{2,\xi}+p}\right) S_X(x)\mathrm{d}x = p.$$

Take the derivative with respect to ξ on both sides of the equation,

$$\frac{\mathrm{d}}{\mathrm{d}\xi}d_{2,\xi} = \frac{S_X\left(a-\xi\right)}{S_X(d_{2,\xi}) - S_X(d_{2,\xi}+p)} \geq 1.$$

It leads to

$$\begin{aligned}
h'(\xi) = & - S_X(a-\xi)u'(a-\xi) \\
& + S_X(d_{2,\xi})u'(d_{2,\xi}-\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}d_{2,\xi} - \int_a^{d_{2,\xi}} S_X(x)u''(x-\xi)dx \\
& - S_X(d_{2,\xi}+p)u'(d_{2,\xi}-\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}d_{2,\xi} - \int_{d_{2,\xi}}^{\infty} S_X(x)u''(x-\xi-p)\mathrm{d}x \\
= & \, S_X(a-\xi)\left(u'(d_{2,\xi}-\xi) - u'(a-\xi)\right) \\
& - \int_a^{d_{2,\xi}} S_X(x)u''(x-\xi)\mathrm{d}x - \int_{d_{2,\xi}+p}^{\infty} S_X(x)u''(x-\xi-p)\mathrm{d}x.
\end{aligned}$$

The second derivative of h is

$$\begin{aligned}
h''(\xi) &= f_X(a-\xi)\left(u'(d_{2,\xi}-\xi)-u'(a-\xi)\right) \\
&\quad + S_X(a-\xi)\left(u''(d_{2,\xi}-\xi)\left(\frac{\mathrm{d}}{\mathrm{d}\xi}d_{2,\xi}-1\right)+u''(a-\xi)\right) \\
&\quad - \left[S_X(d_{2,\xi})u''(d_{2,\xi}-\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}d_{2,\xi}+\int_a^{d_{2,\xi}} S_X(x)u'''(x-\xi)(-1)\mathrm{d}x\right] \\
&\quad - \left[-S_X(d_{2,\xi}+p)u''(d_{2,\xi}-\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}d_{2,\xi}+\int_{d_{2,\xi}+p}^{\infty} S_X(x)u'''(x-\xi-p)(-1)\mathrm{d}x\right] \\
&= f_X(a-\xi)\left(u'(d_{2,\xi}-\xi)-u'(a-\xi)\right)+S_X(a-\xi)\left[u''(a-\xi)-u''(d_{2,\xi}-\xi)\right] \\
&\quad + \left[S_X(a-\xi)-S_X(d_{2,\xi})+S_X(d_{2,\xi}+p)\right]u''(d_{2,\xi}-\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}d_{2,\xi} \\
&\quad + \int_a^{d_{2,\xi}} S_X(x)u'''(x-\xi)\mathrm{d}x+\int_{d_{2,\xi}+p}^{\infty} S_X(x)u'''(x-\xi-p)\mathrm{d}x \\
&= f_X(a-\xi)\left(u'(d_{2,\xi}-\xi)-u'(a-\xi)\right)+S_X(a-\xi)\left[u''(a-\xi)-u''(d_{2,\xi}-\xi)\right] \\
&\quad + \left[S_X(a-\xi)-S_X(d_{2,\xi})+S_X(d_{2,\xi}+p)\right]u''(d_{2,\xi}-\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}d_{2,\xi} \\
&\quad + S_X(d_{2,\xi})u''(d_{2,\xi}-\xi)-S_X(a)u''(a-\xi)+\int_a^{d_{2,\xi}} u''(x-\xi)f_X(x)\mathrm{d}x \\
&\quad - S_X(d_{2,\xi}+p)u''(d_{2,\xi}-\xi)+\int_{d_{2,\xi}+p}^{\infty} u''(x-\xi-p)f_X(x)\mathrm{d}x \\
&= f_X(a-\xi)\left(u'(d_{2,\xi}-\xi)-u'(a-\xi)\right) \\
&\quad + \left[S_X(a-\xi)-S_X(d_{2,\xi})+S_X(d_{2,\xi}+p)\right]u''(d_{2,\xi}-\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}d_{2,\xi} \\
&\quad + \int_a^{d_{2,\xi}} u''(x-\xi)f_X(x)\mathrm{d}x+\int_{d_{2,\xi}+p}^{\infty} u''(x-\xi-p)f_X(x)\mathrm{d}x \\
&\quad + u''(a-\xi)(S_X(a-\xi)-S_X(a)) \\
&\quad - u''(d_{2,\xi}-\xi)\left[S_X(a-\xi)-S_X(d_{2,\xi})+S_X(d_{2,\xi}+p)\right] \\
&= f_X(a-\xi)\left(u'(d_{2,\xi}-\xi)-u'(a-\xi)\right)+u''(a-\xi)(S_X(a-\xi)-S_X(a)) \\
&\quad + u''(d_{2,\xi}-\xi)\left[S_X(a-\xi)-S_X(d_{2,\xi})+S_X(d_{2,\xi}+p)\right]\left(\frac{\mathrm{d}}{\mathrm{d}\xi}d_{2,\xi}-1\right) \\
&\quad + \int_a^{d_{2,\xi}} u''(x-\xi)f_X(x)\mathrm{d}x+\int_{d_{2,\xi}+p}^{\infty} u''(x-\xi-p)f_X(x)\mathrm{d}x \\
&\geq 0,
\end{aligned}$$

where all integrations are finite because the existence assumption of all expectations and $h''(\xi) \geq 0$ is due to convexity of $u(\cdot)$, non-increasing property of survival function $S_X(\cdot)$, $d_{2,\xi} \geq a \geq a - \xi$ and $\frac{d}{d\xi} d_{2,\xi} - 1 \geq 0$. Therefore, h' in non-decreasing in ξ. It is easy to see from the definition of $d_{2,\xi}$ that when $\xi = \xi_2$, we have $d_{2,\xi_2} = a$ and thus

$$h'(\xi_2) = -\int_{a+p}^{\infty} S_X(x) u''(x - \xi - p) \mathrm{d}x < 0.$$

Denote

$$\xi^* \triangleq \sup \left\{ \xi \in [\xi_2, \xi_M] : h'(\xi) < 0 \right\}.$$

If $h'(\xi_M) \leq 0$, then h' is always non-positive for any $\xi \in [\xi_2, \xi_M]$, i.e. $\xi^* = \xi_M$; if $h'(\xi_M) > 0$, then $\xi^* < \xi_M$ and $h'(\xi^*) = 0$.

In summary, on the non-negative real line, $h(\xi)$ is continuous and $h'(\xi) \leq 0$ for $\xi \in [\xi_0, \xi^*]$ and $h'(\xi) \geq 0$ for $\xi \in [\xi^*, \xi_M]$. Therefore, $h(\xi)$ achieves it minimal value at ξ^* and the reinsurance contracts $I^*_{\xi^*}$ summarized in the theorem is the optimal solutions of the two-step minimization problem (2.13), i.e.

$$\min_{\xi_0 \leq \xi \leq \xi_M} \left\{ \min_{I \in \mathcal{I}_{p,0},\, I(a) = \xi} H(I) \right\} = H(I^*_{\xi^*}).$$

Since $\xi^* \in [\xi_0, \xi_M]$, the corresponding contract $I^*_{\xi^*}$ is in $\mathcal{I}_{p,0}$. Thus,

$$H(I^*_{\xi^*}) \geq \min_{I \in \mathcal{I}_{p,0}} H(I),$$

and furthermore

$$\min_{\xi_0 \leq \xi \leq \xi_M} \left\{ \min_{I \in \mathcal{I}_{p,0},\, I(a) = \xi} H(I) \right\} \geq \min_{I \in \mathcal{I}_{p,0}} H(I). \tag{2.34}$$

On the other hand, for an arbitrary $k \in \mathcal{I}_{p,0}$,

$$H(k) \geq \min_{I \in \mathcal{I}_{p,0},\, I(a) = k(a)} H(I) \geq \min_{\xi_0 \leq \xi \leq \xi_M} \left\{ \min_{I \in \mathcal{I}_{p,0},\, I(a) = \xi} H(I) \right\}.$$

Going through all contracts in $\mathcal{I}_{p,0}$, we have

$$\min_{k \in \mathcal{I}_{p,0}} H(k) \geq \min_{\xi_0 \leq \xi \leq \xi_M} \left\{ \min_{I \in \mathcal{I}_{p,0},\, I(a) = \xi} H(I) \right\}. \tag{2.35}$$

Combining inequalities (2.34) and (2.35), we conclude that

$$\min_{I \in \mathcal{I}_{p,0}} H(I) = \min_{\xi_0 \le \xi \le \xi_M} \left\{ \min_{I \in \mathcal{I}_{p,0},\, I(a)=\xi} H(I) \right\},$$

and $I^*_{\xi^*}$ is also the optimal solution to Problem (2.6). ■

Proof of Lemma 2.2.1. For a feasible reinsurance contract $I \in \mathcal{I}$, choosing two particular points $d_1 = a \wedge b - I(a \wedge b)$ and $d_2 = a \vee b - (I(a \vee b) - I(a \wedge b))$ and substituting them into expression (2.23), then it is not hard to see $m_I(a) = I(a)$, $m_I(b) = I(b)$ and $m_I(x) \le I(x)$ for all $x \ge 0$. Moreover, $P_{m_I} \le P_I$. Denote $\xi_a \triangleq I(a) = m_I(a)$ and $\xi_b \triangleq I(b) = m_I(b)$. Then, $V(I) = b - \xi_b \wedge (\xi_a + P_I) + P_I$ and $V(m_I) = b - \xi_b \wedge (\xi_a + P_{m_I}) + P_{m_I}$.

(1) Suppose $\alpha \ge \beta$ (or equivalently $b \ge a$). In this case, $\xi_b \le \xi_a \le \xi_a + P_I \wedge P_{m_I}$. It implies that

$$V(I) - V(m_I) = (b - \xi_b + P_I) - (b - \xi_b + P_{m_I}) = P_I - P_{m_I} \ge 0.$$

(2) Suppose $\alpha \ge \beta$ (or equivalently $b \ge a$). In this case, $\xi_a \le \xi_b$ and

$$\begin{aligned} V(I) - V(m_I) &= (b - \xi_b \wedge (\xi_a + P_I) + P_I) - (b - \xi_b \wedge (\xi_a + P_{m_I}) + P_{m_I}) \\ &= \xi_b \wedge (\xi_a + P_{m_I}) - \xi_b \wedge (\xi_a + P_I) + P_I - P_{m_I}. \end{aligned}$$

Furthermore, if $\xi_b \le \xi_a + P_{m_I}$, we have $\xi_b \le \xi_a + P_{m_I} \le \xi_a + P_I$ and thus

$$V(I) - V(m_I) = \xi_b - \xi_b + P_I - P_{m_I} \ge 0.$$

If $\xi_a + P_{m_I} < \xi_b \le \xi_a + P_I$, then

$$V(I) - V(m_I) = \xi_a + P_{m_I} - \xi_b + P_I - P_{m_I} = P_I + \xi_a - \xi_b \ge 0.$$

If $\xi_a + P_I < \xi_b$, we have $\xi_a + P_{m_I} \le \xi_a + P_I < \xi_b$ and thus

$$V(I) - V(m_I) = (\xi_a + P_{m_I}) - (\xi_a + P_I) + P_I - P_{m_I} = 0.$$

In short, when $\alpha \ge \beta$, we have $V(I) - V(m_I) \ge 0$.

Combining these two cases, we get the result as desired. ■

Proof of Theorem 2.2.2. For each $d_1 \in [0,\, a \wedge b]$, define $v_{d_1}(d_2) = v(d_1, d_2)$ as the function of $d_2 \in [a \vee b,\, \infty)$.

(1) Suppose $\alpha \leq \beta$ (or equivalently $b \leq a$). In this case, for any fixed $d_1 \in [0, b]$,and for any $d_2 \in [b, a]$ and I given by (2.23), we have

$$v_{d_1}(d_2) \triangleq v(d_1, d_2) = v(I) = b - (b - d_1) + P_I = d_1 + (1+\theta)\left(\int_{d_1}^{b} + \int_{d_2}^{a}\right) S_X(x)\mathrm{d}x,$$

Clearly, the first derivative of $v_{d_1}(d_2)$ satisfies

$$v'_{d_1}(d_2) = \frac{\partial}{\partial d_2} v(d_1, d_2) = -\,(1+\theta)\, S_X(d_2) < 0.$$

Thus, for any $d_1 \in [0, b]$, $d_2^*(d_1) = a$ is the minimizer to the minimization problem $\min\limits_{d_2 \in [b,a]} v(d_1, d_2)$ of (2.25). Hence,

$$\min_{d_1 \in [0,b]} \min_{d_2 \in [b,a]} v(d_1, d_2) = \min_{d_1 \in [0,b]} v(d_1, d_2^*(d_1)) = \min_{d_1 \in [0,b]} v(d_1, a). \tag{2.36}$$

Next, we consider the function

$$v(d_1, d_2^*(d_1)) = v(d_1, a) = d_1 + (1+\theta)\int_{d_1}^{b} S_X(x)\mathrm{d}x.$$

Obviously, the function $v(d_1, d_2^*(d_1))$ is continuous in d_1 and its first derivative is $\frac{\mathrm{d}}{\mathrm{d}d_1} v(d_1, d_2^*(d_1)) = 1 - (1+\theta)\, S_X(d_1)$. Since

$$d_1 \leq \mathrm{VaR}_{\frac{1}{1+\theta}}(X) \Leftrightarrow S_X(d_1) \geq \frac{1}{1+\theta},$$

thus, $d_1^* = \mathrm{VaR}_{\frac{1}{1+\theta}}(X) \wedge b$ is the minimizer to the minimization problem $\min\limits_{d_1 \in [0,b]} v(d_1, a)$ of (2.36). It follows that the optimal contract I^* has the form (2.23) with $d_1 = d_1^*$ and $d_2 = d_2^*\left(d_1^*\right) = a$. Namely, we have

$$I^*(x) \quad = \quad (x - \mathrm{VaR}_{\frac{1}{1+\theta}}(X) \wedge b)^+ - (x-b)^+.$$

(2) Suppose $\alpha \geq \beta$ (or equivalently $b \geq a$) and $\alpha \leq \frac{1}{1+\theta}$. In this case, for any fix $d_1 \in [0, a]$, for any $d_2 \in [a, b]$,

$$\begin{aligned} v_{d_1}(d_2) &= v(d_1, d_2) = V(m_I) = b - (a - d_1 + b - d_2) \wedge (a - d_1 + P_I) + P_I \\ &= \begin{cases} -a + d_1 + d_2 + P_I, & \text{if} \quad b \leq P_I + d_2; \\ -a + d_1 + b, & \text{if} \quad b > P_I + d_2. \end{cases} \end{aligned}$$

Moreover, define $G_{d_1}(d_2)$ as

$$G_{d_1}(d_2) \triangleq P_I - (b - d_2) = (1+\theta)\left(\int_{d_1}^{a} + \int_{d_2}^{b}\right) S_X(x)\mathrm{d}x - b + d_2.$$

In order to determine the sign of $G_{d_1}(d_2)$, we need to consider its monotonicity on $[a,b]$. It is not hard to see $G'_{d_1}(d_2) = 1 - (1+\theta)S_X(d_2) \geq 0$ for any $d_2 \in [a,b]$ because $\alpha \leq \frac{1}{1+\theta}$ or $a \geq \mathrm{VaR}_{\frac{1}{1+\theta}}(X)$. Thus, G_{d_1} is a continuous and non-decreasing function of d_2 on $[a,b]$. If $G_{d_1}(a) \geq 0$, then $G_{d_1} \geq 0$ on the interval $[a,b]$. If $G_{d_1}(a) \leq 0$, there exists $c(d_1) \in [a,b]$ such that $G_{d_1}(d_2) \leq 0$ for any $d_2 \in [a, c(d_1)]$ and $G_{d_1}(d_2) \geq 0$ for any $d_2 \in [c(d_1), b]$. Thus, to determine the optimal solution I^*, we need to consider the following three cases.

Case 1. Suppose $(1+\theta)\int_0^b S_X(x)\mathrm{d}x - b + a \leq 0$.
In this case, $G_{d_1}(a) \leq 0$ for any $d_1 \in [0,a]$. Thus,

$$v_{d_1}(d_2) = \begin{cases} -a + d_1 + b, & \text{for } a \leq d_2 \leq c(d_1); \\ -a + d_1 + d_2 + P_I, & \text{for } c(d_1) \leq d_2 \leq b; \end{cases}$$

with non-negative first derivative

$$v'_{d_1}(d_2) = \begin{cases} 0, & \text{for } a \leq d_2 \leq c(d_1); \\ 1 - (1+\theta)S_X(d_2), & \text{for } c(d_1) \leq d_2 \leq b. \end{cases}$$

Thus $d_2^*(d_1) = a$ and then $v(d_1, d_2^*(d_1)) = v(d_1, a) = -a + d_1 + b$ is a continuous function of d_1. It implies that

$$\begin{aligned} \min_{(d_1,d_2)\in[0,a]\times[a,b]} v(d_1,d_2) &= \min_{d_1\in[0,a]}\left\{\min_{d_2\in[a,b]} v(d_1,d_2)\right\} \min_{d_1\in[0,a]} v(d_1, d_2^*(d_1)) \\ &= \min_{d_1\in[0,a]} -a + d_1 + b = b - a, \end{aligned}$$

namely, the optimal pair is $(d_1^*, d_2^*) = (0, a)$. The corresponding optimal contract is $I^*(x) = x - (x-b)^+$

Case 2. Suppose $(1+\theta)\int_a^b S_X(x)\mathrm{d}x - b + a \geq 0$. In this case, $G_{d_1}(a) \geq 0$ for any $d_1 \in [0,a]$. Thus $v_{d_1}(d_2) = -a + d_1 + d_2 + P_I$, with non-positive first derivative $v'_{d_1}(d_2) = 1 - (1+\theta)S_X(d_2)$. It implies that $d_2^*(d_1) = a$ for any $d_1 \in [0,a]$. Now, $v(d_1, d_2^*(d_1)) = v(d_1, a) = d_1 + P_I$ is a continuous function of d_1, then

$$\min_{(d_1,d_2)\in[0,a]\times[a,b]} v(d_1,d_2) = \min_{d_1\in[0,a]}\left\{\min_{d_2\in[a,b]} v(d_1,d_2)\right\} = \min_{d_1\in[0,a]} v(d_1, d_2^*(d_1)).$$

Note that $\frac{\mathrm{d}}{\mathrm{d}\,d_1}v(d_1,d_2^*(d_1)) = 1-(1+\theta)S_X(d_1)\le 0$ is equivalent to $d_1 \le \mathrm{VaR}_{\frac{1}{1+\theta}}(X)$ and that $\mathrm{VaR}_{\frac{1}{1+\theta}}(X)\le a$ by assumption. Thus, the optimal pair is $(d_1^*,d_2^*) = (\mathrm{VaR}_{\frac{1}{1+\theta}(X)}, a)$ and the corresponding optimal contract is $I^*(x) = (x-\mathrm{VaR}_{\frac{1}{1+\theta}(X)})^+ - (x-b)^+$.

Case 3. Suppose $(1+\theta)\int_a^b S_X(x)\mathrm{d}x - b + a \le 0 \le (1+\theta)\int_0^b S_X(x)\mathrm{d}x - b + a$. In this case, there exists $d_0\in[0,a]$ satisfying $G_{d_1}(a) = (1+\theta)\int_{d_1}^b S_X(x)dx - b + a \ge 0$ for $0\le d_1\le d_0$ and $G_{d_1}(a)\le 0$ for $d_0\le d_1\le a$. For $d_1\in[0,d_0]$, i.e. $G_{d_1}(a)\ge 0$, one has $v_{d_1}(d_2^*(d_1)) = d_1 + P_I$ by the same argument as in Case 2. For $d_1\in[d_0,a]$, i.e. $G_{d_1}(a)\le 0$ one has $v_{d_1}(d_2^*(d_1)) = -a+d_1+b$ by the same argument as in Case 1. Thus, on the whole interval $[0,a]$, $v_{d_1}(d_2^*(d_1)) = d_1 + P_I\vee(b-a)$ is continuous of d_1 and its first derivative is

$$\frac{\mathrm{d}}{\mathrm{d}d_1}v_{d_1}(d_2^*(d_1)) = \begin{cases} 1-(1+\theta)S_X(d_1), & \text{for } 0\le d_1\le d_0;\\ 1, & \text{for } d_0\le d_1\le a.\end{cases}$$

It implies that

$$\min_{(d_1,d_2)\in[0,a]\times[a,b]} v(d_1,d_2) = \min_{d_1\in[0,a]}\left\{\min_{d_2\in[a,b]} v(d_1,d_2)\right\} = \min_{d_1\in[0,a]} v(d_1,d_2^*(d_1)) = v(d_1^*,d_2^*(d_1^*)),$$

where $d_1^* = \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\wedge d_0$. The corresponding optimal solution is $I^*(x) = (x-d_1^*)^+ - (x-b)^+$.

Combining these three cases, the optimal solution of Problem (2.22) can be summarized into a unified formula which is a limited stop-loss reinsurance contract given as follows:

$$I^*(x) = (x-\max\{0, d^*\wedge \mathrm{VaR}_{\frac{1}{1+\theta}}(X)\})^+ - (x-b)^+,$$

where $d^*\in\mathbb{R}$ is the solution of the equation $(1+\theta)\int_{d^*}^b S_X(x)\mathrm{d}x = b-a$. ■

Chapter 3

Convex Risk Measure and Wang's Premium Principle

In this chapter, we consider a general framework of the optimal reinsurance design problem from the perspective of the insurer. In a reinsurance market, there are often more than one available reinsurer with different pricing schemes. By ceding its loss X to n competitive reinsurers, the insurance company may pay a smaller premium. It is natural to consider the optimal reinsurance model with multiple reinsurers who may have different risk attitudes. In the case when the insurer shares the loss X with n reinsurers, by ceding $I_i(X)$ to Reinsurer i, $i = 1, \ldots, n$, the retained loss for the insurer is $R(X) = X - I(X)$ where $I(X) = \sum_{i=1}^{n} I_i(X)$ is the total ceded loss. The total premium for the insurer is $P_I = \sum_{i=1}^{n} P_{i,I_i}$ where P_{i,I_i} is the premium for the contract I_i, $i = 1, \ldots, n$. Since the underlying loss X is splitting into $n+1$ components, $I_i(X)$ for $i = 1, \ldots, n$ and $R(X)$, each ceded loss function I_i, $i = 1, \ldots, n$ and retained loss function R should be feasible. In other words, a group of reinsurance contracts $(I_1, \ldots, I_n)$ is called "*feasible*" if $R \in \mathcal{I}$ and $I_i \in \mathcal{I}$, $i = 1, \ldots, n$. We shall denote

$$\mathcal{I}^n = \{(I_1, \ldots, I_n) : I_i \in \mathcal{I} \text{ for } i = 1, \ldots, n, \text{ and } R \in \mathcal{I}\} \tag{3.1}$$

to be the set of all feasible groups $(I_1, \ldots, I_n)$.

As counterparties in one reinsurance contract I, the insurer and the reinsurers adopt risk measure principles based on their own risk attitudes which are generally not the same. In our work, we assume the insurer uses a convex risk measure ρ given by Definition 1.2.10 while the i-th reinsurer uses Wang's premium principle with a distortion function $g_i(\cdot)$ given by Definition 1.3.4. Instead of assigning a particular risk measure, here by only assuming a convex risk measure and Wang's premium principle, we use a very general framework based on families of risk measures.

Throughout this chapter, we denote as $\mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ the set of all bounded random variables on an atom-less probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume the insurer faces a non-negative insurable risk $X \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$, and the survival function $S_X(x)$ of X is assumed to be continuous and strictly decreasing on $[0, \bar{X}]$. Therefore, the essential supremum of X is a finite value, i.e.

$$\bar{X} = \operatorname{ess\,sup} X = \inf\{a \in \mathbb{R} : \mathbb{P}(X > a) = 0\} < \infty,$$

and the support of X is the closed interval $[0, \bar{X}]$. With respect to X, the set of all feasible reinsurance contracts, which is denoted by $\mathcal{I}$, contains all non-decreasing and 1-Lipschitz continuous functions defined on $[0, \bar{X}]$.

In the present work, we consider the optimization problem from the insurer's point of view. Mathematically speaking, the problem can be formulated as follows:

$$\min_{(I_1,\ldots,I_n)\in\mathcal{I}^n} \rho\left(X - \sum_{i=1}^{n} I_i(X) + \sum_{i=1}^{n} P_{i,I_i}\right), \tag{3.2}$$

where ρ is a convex risk measure chosen by the insurer, while for each $i = 1, \ldots, n$,

$$P_{i,I_i} = \int_0^\infty g_i \circ S_{I_i(X)}(t)\mathrm{d}t,$$

follows Wang's premium principle with respect to the distortion function g_i.

3.1 Reinsurance Model with Single Reinsurer

In the first place, we consider the classical single reinsurer model, that is, the insurer purchasing a reinsurance strategy, denoted by the function $I(x)$, from the reinsurer by paying the premium P_I which follows the Wang's premium principle. The insurer's retained loss becomes $R(x) \triangleq x - I(x)$, and we are looking for the optimal strategy $I \in \mathcal{I}$ for the insurer such that the insurer's total loss, which is $R(X) + P_I$, is minimized under a law-invariant convex risk measure ρ. Mathematically speaking, we are considering the following problem:

$$\min_{I\in\mathcal{I}} \rho\left(X - I(X) + P_I\right) \tag{3.3}$$

$$\text{such that } P_I = \int_0^{\bar{X}} g \circ S_{I(X)}(t)\mathrm{d}t,$$

where $g : [0, 1] \to [0, 1]$ is assumed to be a strictly concave, twice differentiable and non-decreasing distortion function.

In Problem (3.3), both the risk measure and the premium should be viewed as generalized expressions and represent a family. Reinsurers with different risk bearings will choose different distortions g and this will lead to different premium values for the same reinsurance contract I. Similarly, insurers take different risk measures ρ based on their own preference. A lot of research has been done when one of them is restricted to a particular choice while the other one is given by the generalized expression. For example, [Chi and Tan, 2013] considered the coherent risk measure AVaR and solved the following problem

$$\begin{aligned} &\min_{I \in \mathcal{I}} \mathrm{AVaR}_\alpha(X - I(X) + P_I) \\ &\text{such that } P_I = \int_0^{\bar{X}} g \circ S_{I(X)}(t)\mathrm{d}t. \end{aligned}$$

[Cheung et al., 2014] chose the expectation pricing principle, i.e. $g(x) = x$ which leads to the problem

$$\begin{aligned} &\min_{I \in \mathcal{I}} \rho\left(X - I(X) + P_I\right) \\ &\text{such that } P_I = \mathbb{E}[I(X)]. \end{aligned}$$

In our formulation, without further assumption, we do not assume a particular risk measure/premium principle has been chosen by the insurer/reinsurer. We would like to provide a general formula for the optimal reinsurance contract to Problem (3.3) that can be applied to any particular law-invariant convex measure and Wang's premium. To solve Problem (3.3), we are going to use the equivalent expression of a law-invariant convex risk measure introduced in Lemma 1.2.2, namely,

$$\rho(X) = \sup_{\mu \in \mathcal{P}([0,1])} \left\{ \int_0^1 \mathrm{AVaR}_\alpha(X)\mu(\mathrm{d}\alpha) - \beta(\mu) \right\}, \tag{3.4}$$

where $\beta : \mathcal{P}([0, 1]) \to [0, \infty]$ is a law invariant, lower semi-continuous and convex function.

For an arbitrary selected law-invariant convex risk measure ρ, the existence of the optimal solution to Problem (3.3) is not guaranteed. Therefore, at this moment, we consider the infimum value of $\rho\left(X - I(X) + P_I\right)$ among the set $\mathcal{I}$, that is

$$\inf_{I \in \mathcal{I}} \rho\left(X - I(X) + P_I\right). \tag{3.5}$$

After that, we will show the equivalence between Problem (3.3) and Problem (3.5).

Lemma 3.1.1 *Problem (3.5) has the following minimax expression:*

$$\inf_{I\in\mathcal{I}} \sup_{\mu\in\mathcal{P}([0,1])} f(I,\mu), \tag{3.6}$$

where the function $f : \mathcal{I} \times \mathcal{P}([0,1]) \to \mathbb{R}$ is defined via

$$f(I,\mu) \triangleq \int_0^1 \mathrm{AVaR}_\alpha(X - I(X))\mu(\mathrm{d}\alpha) + \int_0^{\bar{X}} g \circ S_{I(X)}(t)\,\mathrm{d}t - \beta(\mu). \tag{3.7}$$

Lemma 3.1.1 translates Problem (3.5) to the minimax problem (3.6), which can be solved with the help of the following useful minimax theorem given by [Fan, 1953] and references therein.

Theorem 3.1.2 (Minimax Theorem) *Let Ξ_1 be a non-empty compact convex Hausdorff topological vector space, and Ξ_2 be a non-empty convex set. Let f be a real-valued function defined on $\Xi_1 \times \Xi_2$ such that*

1) $\xi_1 \mapsto f(\xi_1,\xi_2)$ is convex and lower-semicontinuous on Ξ_1 for each $\xi_2 \in \Xi_2$;

2) $\xi_2 \mapsto f(\xi_1,\xi_2)$ is concave on Ξ_2 for each $\xi_1 \in \Xi_1$.

Then

$$\inf_{\xi_1\in\Xi_1} \sup_{\xi_2\in\Xi_2} f(\xi_1,\xi_2) = \sup_{\xi_2\in\Xi_2} \inf_{\xi_1\in\Xi_1} f(\xi_1,\xi_2). \tag{3.8}$$

Remark 3.1.1 *If the equation (3.8) holds, the value in (3.8) is called the saddle-value in the minimax problem. A pair $(\xi_1^*,\xi_2^*) \in \Xi_1 \times \Xi_2$ is called a saddle-point of f with respect to $\Xi_1 \times \Xi_2$, if it satisfies*

$$\inf_{\xi_1\in\Xi_1} f(\xi_1,\xi_2^*) = \sup_{\xi_2\in\Xi_2} f(\xi_1^*,\xi_2).$$

For an arbitrary real-valued function f defined on the space $\Xi_1 \times \Xi_2$, the equation (3.8) may not hold, although it is always true, as is easily seen, that

$$\inf_{\xi_1\in\Xi_1} \sup_{\xi_2\in\Xi_2} f(\xi_1,\xi_2) \geq \sup_{\xi_2\in\Xi_2} \inf_{\xi_1\in\Xi_1} f(\xi_1,\xi_2). \tag{3.9}$$

The existence of a saddle-point implies the existence of the saddle value. Indeed, the equation (3.8) *is implied by* (3.9) *together with the following observation*

$$\inf_{\xi_1\in\Xi_1}\sup_{\xi_2\in\Xi_2} f(\xi_1,\xi_2) \le \sup_{\xi_2\in\Xi_2} f(\xi_1^*,\xi_2) = \inf_{\xi_1\in\Xi_1} f(\xi_1,\xi_2^*) \le \sup_{\xi_2\in\Xi_2}\inf_{\xi_1\in\Xi_1} f(\xi_1,\xi_2).$$

However, in contrary, the existence of a saddle-value is not a sufficient condition for the existence of a saddle-point.

The Minimax Theorem 3.1.2 will be used in the proof of the following theorem. For each $\mu \in \mathcal{P}([0,1])$, define the following notation:

1. $\phi_\mu \triangleq h_\mu \wedge g$;
2. $G_\mu \triangleq \{t \ge 0 : g \circ S_X(t) < h_\mu \circ S_X(t)\}$;
3. $E_\mu \triangleq \{t \ge 0 : g \circ S_X(t) = h_\mu \circ S_X(t)\}$.

Moreover, define $I_\mu(t)$ to be the reinsurance policy satisfying the following conditions:

$$I_\mu(0) = 0 \text{ and } I'_\mu(t) = \mathbb{I}_{\{G_\mu\}}(t), \text{ for any } t \ge 0, \tag{3.10}$$

where $\mathbb{I}_{\{G_\mu\}}$ is the indicator function associated with the set G_μ, i.e.

$$\mathbb{I}_{\{G_\mu\}}(t) = \begin{cases} 1, & t \in G_\mu, \\ 0, & t \notin G_\mu, \end{cases} \qquad \text{for any } t \ge 0.$$

Theorem 3.1.3 *For a bounded risk variable X, the minimax problem (3.6) has a saddle-value*

$$S \triangleq \sup_{\mu\in\mathcal{P}([0,1])} \left\{ \int_0^\infty \phi_\mu \circ S_X(t)\mathrm{d}t - \beta(\mu) \right\}. \tag{3.11}$$

Moreover, there exists $\mu_0 \in \mathcal{P}([0,1])$ such that $S = \int_0^\infty \phi_{\mu_0} \circ S_X(t)\mathrm{d}t - \beta(\mu_0)$.

Remark 3.1.2 *In the proof of Theorem 3.1.3 given in Section 3.3, the Minimax Theorem 3.1.2 is used to exchange the order of minimum and supremum in the expression* (3.6)*. In order to guarantee the compactness of the set $\mathcal{I}$, which is required by the Minimax Theorem 3.1.2, we need to assume that X is a bounded random loss, i.e. $X \in \mathcal{L}^\infty(\Omega,\mathbb{P},\mathcal{F})$.*

Theorem 3.1.3 shows that Problem (3.5) has the infimum value

$$\int_0^{\bar{X}} \phi_{\mu_0} \circ S_X(t)\mathrm{d}t - \beta(\mu_0). \tag{3.12}$$

However, as mentioned in Remark 3.1.1, the existence of S, which is the saddle-value of the function f, is only a necessary but not sufficient condition for the existence of the saddle-point. Therefore, the result of Theorem 3.1.3 is not enough to determine the optimal reinsurance contract that leads to this minimal value.

First, we need to show that the infimum value S is indeed the minimum value for Problem (3.3). Note that, Problem (3.3) and Problem (3.5) are equivalent if and only if the optimal solution to Problem (3.3) exists. This existence is given in the following theorem.

Theorem 3.1.4 *Problem (3.3) is well-defined in the sense that there exists an optimal solution $I^* \in \mathcal{I}$. Therefore, the infimum value S given in (3.11) is also the minimal value of Problem (3.3) and it can be achieved at I^*.*

The next proposition provides a necessary condition for the expression of the minimizer of Problem (3.3).

Proposition 3.1.5 *Assume $\mu_0 \in \mathcal{P}([0,1])$ is given by (3.12), i.e. $S = f(I_{\mu_0}, \mu_0)$, where S is defined by (3.11). Then, any optimal solution I_0 for Problem (3.3) must satisfy the following conditions:*

$$I_0(0) = 0 \text{ and } I_0'(t) = \mathbb{I}_{\{G_{\mu_0}\}}(t) + \alpha(t)\mathbb{I}_{\{E_{\mu_0}\}}(t), \text{ for any } t \geq 0, \tag{3.13}$$

where $\alpha(t)$ is some function between $[0,1]$.

The following two theorems give the expressions of the optimal reinsurance contract under two particular cases.

Definition 3.1.1 *(1) A pair X and Y of random variables is comonotone on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if*

$$\left(X(\omega_1) - X(\omega_2)\right)\left(Y(\omega_1) - Y(\omega_2)\right) \geq 0 \text{ almost surely w.r.t. } \mathbb{P} \text{ on } \Omega.$$

(2) A risk measure $\rho : \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R} \to \mathbb{R}$ is comonotone, if

$$\rho(X+Y) = \rho(X) + \rho(Y), \text{ for any comonotone pair } X, Y \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}).$$

Theorem 3.1.6 *Suppose $\rho : \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is a law-invariant and comonotone coherent risk measure. Then, there exists a probability measure μ on $[0,1]$ such that any reinsurance contract of the form*

$$I_0(x) = \int_0^x \left[\mathbb{I}_{\{G_\mu\}}(t) + \alpha(t)\mathbb{I}_{\{E_\mu\}}(t)\right] \mathrm{d}t, \text{ for any } t \geq 0, \tag{3.14}$$

where $\alpha(t)$ is an arbitrary function between $[0,1]$, will be a minimizer of Problem (3.3).

Theorem 3.1.7 *In addition to the assumptions of Proposition 3.1.5, if the set E_{μ_0} has Lebesgue measure zero, then the reinsurance contract*

$$I_{\mu_0}(x) \triangleq \int_0^x \mathbb{I}_{\{G_{\mu_0}\}}(t)\,\mathrm{d}t, \text{ for } x \geq 0,$$

defined by (3.10) is one optimal solution to Problem (3.3). Moreover, the corresponding minimal value is

$$\min_{I\in\mathcal{I}} \rho\left(X - I(X) + P_I\right) = \int_0^\infty \phi_{\mu_0} \circ S_X(t)\mathrm{d}t - \beta(\mu_0).$$

Remark 3.1.3 *According to Theorem 3.1.7, the necessary condition for optimality of reinsurance contract given by expression (3.13) is also a sufficient condition. It implies that (I_{μ_0}, μ_0) is a saddle point of the minimax function $f(I,\mu)$ on $\mathcal{I} \times \mathcal{P}([0,1])$, i.e.*

$$\sup_{\mu\in\mathcal{P}([0,1])} f(I_{\mu_0}, \mu) = f(I_{\mu_0}, \mu_0) = \min_{I\in\mathcal{I}} f(I, \mu_0). \tag{3.15}$$

Remark 3.1.4 *Theorem 3.1.7 is a consequence of Theorem 3.1.4 and Proposition 3.1.5. Alternatively, Theorem 3.1.7 can be proved directly from Theorem 3.1.3 by using the argument involving direction derivative. This proof is summarized at the end of the appendix of this chapter.*

Example 3.1.1 *A very commonly used coherent risk measure, thus convex risk measure, is the Average Value-at-Risk (AVaR). Given a level α, one can define a convex function $\beta : \mathcal{P}([0,1]) \to \mathbb{R} \cup \{+\infty\}$ via $\beta(\mu) = 0$, if $\mu = \delta_\alpha$, otherwise $\beta(\mu) = +\infty$. Then, AVaR_α can be induced by substituting β into Definition 3.4 for the convex risk measure.*

Since β only takes a finite value at δ_α, the function

$$\int_0^{\bar{X}} \phi_\mu \circ S_X(t)\mathrm{d}t - \beta(\mu)$$

achieves its maximal value at probability measure δ_α, i.e. $\mu_0 = \delta_\alpha$. Moreover, $h'_{\delta_\alpha}(s) = \frac{1}{\alpha}\mathbb{I}_{\{[0,\alpha]\}}$ and thus $h_{\delta_\alpha}(x) = \frac{1}{\alpha}x\mathbb{I}_{\{[0,\alpha]\}} + \mathbb{I}_{\{(\alpha,1]\}}$. Note that, functions h_{δ_α} and g will cross at most once. When $g'(0) > \frac{1}{\alpha}$, they do cross, and denote by d^* the root of equation

$$\frac{1}{\alpha} = \frac{g(S_X(d^*))}{S_X(d^*)},$$

where $a = VaR_\alpha(X)$; when $g'(0) \leq \frac{1}{\alpha}$, g is always smaller or equal to h_{δ_α} on $[0,1]$ and we use $d^* = \bar{X}$ in this case. It can be easily checked that $VaR_\alpha(X) \leq d^*$ and $G_{\delta_\alpha} = [d^*, \bar{X}]$. Therefore, Theorem 3.1.6 says that the optimal solution to Problem (3.3), by using the expression (3.14), is

$$I^*(x) = x - (x - d^*)^+, \tag{3.16}$$

and the corresponding minimal value is

$$\begin{aligned}\min_{I\in\mathcal{I}} \rho\left(X - I(X) + P_I\right) &= \int_0^{\bar{X}} \phi_{\delta_\alpha} \circ S_X(t)\mathrm{d}t - \beta(\delta_\alpha) \\ &= a + \int_{d^*}^{\bar{X}} S_X(t)\mathrm{d}t + \int_0^{d^*} g \circ S_X(t)\mathrm{d}t.\end{aligned}$$

This result is consistent with the known result, see details in [Chi and Tan, 2013].

Example 3.1.2 *Suppose the reinsurer adopts the net premium principle, which is the actuarial premium principle with zero risk loading, i.e. $P_I = \mathbb{E}[I(X)]$. In this case, the premium could be viewed as a Wang's premium with linear distortion $g(x) = x$. For any probability measure μ, the induced concave function h_μ is no less than g on the entire interval $[0,1]$ and thus $\phi_\mu = h_\mu \wedge g = g$. It implies that $G_{\mu_0} = [0,1]$. Therefore, $I_{\mu_0}(x) = x$ on $[0,1]$, i.e. the optimal reinsurance contract is the full reinsurance. This result is consistent with what is showed in [Cheung et al., 2014].*

3.2 Reinsurance Model with Multiple Reinsurers

In this section, we discuss the optimal reinsurance problem (3.2) in the multiple reinsurers case with the help of results obtained in Section 3.1.

Recall that, in the n-reinsurer model, the ceded loss function or indemnity function for Insurer i is denoted by I_i, which is assumed to satisfies $I_i \in \mathcal{I}$, and the corresponding premium is P_{i,I_i}, $i = 1, \dots, n$. Then, the total ceded loss is $I(X) = \sum_{i=1}^{n} I_i(X)$, while the loss retained to the insurer is $R(X) = X - I(X)$. The optimal reinsurance problem we are considering is

$$\min_{(I_1,\dots,I_n)\in\mathcal{I}^n} \rho\left(X - I(X) + \sum_{i=1}^{n} P_{i,I_i}\right) \tag{3.17}$$
$$\text{such that } P_{i,I_i} \triangleq \int_0^{\bar{X}} g_i \circ S_{I_i(X)}(t)\,\mathrm{d}t, \text{ for } i = 1, \dots, n;$$

where $g_i : [0, 1] \to [0, 1]$, $i = 1, \dots, n$ are continuous, twice differentiable, non-decreasing and concave distortion functions, and $(I_1, \dots, I_n) \in \mathcal{I}^n$ means

1. $I_i : [0, \infty) \to [0, \infty)$ satisfy $I_i(0) = 0$ and I_i is non-decreasing, $i = 1, \dots, n$;
2. $0 \leq I_i(y) - I_i(x) \leq y - x$, for any $0 \leq x \leq y < \infty$, $i = 1, \dots, n$;
3. $0 \leq R(y) - R(x) \leq y - x$, for any $0 \leq x \leq y < \infty$.

From the properties of g_i, $i = 1, \dots, n$, the function $g : [0, 1] \to [0, 1]$ defined as follows

$$g(t) \triangleq \min\{g_i(t), i = 1, \dots, n\} = g_1(t) \wedge \cdots \wedge g_n(t), \quad \forall t \geq 0,$$

is a continuous, twice differentiable, non-decreasing and concave function. Therefore, g can be used as a distortion function to define a corresponding Wang's premium P_I which is subadditive:

$$P_I \triangleq \int_0^{\bar{X}} g \circ S_{I(X)}(t)\mathrm{d}t \tag{3.18}$$

Denote sets, $i = 1, \dots, n$,

$$A_i \triangleq \{t \geq 0 : g_i \circ S_X(t) = g \circ S_X(t) < g_i \circ S_X(t) \text{ for } j = i + 1, \dots, n\}. \tag{3.19}$$

Lemma 3.2.1 *Problem (3.17) has the same minimal value as the following minimization problem:*

$$\min_{I\in\mathcal{I}} \rho\left(X - I(X) + P_I\right), \tag{3.20}$$

where P_I is defined by (3.18).

Remark 3.2.1 *Since $g(t) = g_1(t) \wedge \cdots \wedge g_n(t) \leq g_i(t)$ for all $t \geq 0$, $i = 1, \ldots, n$, then for any feasible reinsurance contract $I \in \mathcal{I}$ we have $P_I \leq P_{i,I}$, $i = 1, \ldots, n$. Lemma 3.2.1 implies that, from the insurer's point of view, if the insurer is looking for a total coverage $I(X)$ for his underlying loss X, by selecting carefully a portfolio of n reinsurances $(I_1, \ldots, I_n)$ from $\mathcal{I}^n$, the insurer can obtain the same effect as I in the sense that $\sum_{i=1}^n I_i = I$ but only pays a premium $P_I = \sum_{i=1}^n P_{i,I_i}$, which is smaller than the premium $P_{i,I}$ for buying I from Reinsurer i only, $i = 1, \ldots, n$. Therefore, compare to buying one reinsurance contract from a single reinsurer, setting up a portfolio of reinsurance contracts can reduce the total premium, and it can be viewed as a better choice for the insurer based on the concern of premium budget.*

The 1-reinsurer minimization problem (3.20) has been discussed in Section 3.1 and we have found the formula for the minimizer. By using results in Section 3.1, we can now state the main result for the multiple reinsurer model.

Proposition 3.2.2 *Problem (3.17) has the minimal value*

$$\int_0^{\bar{X}} [h_{\mu_0} \circ S_X(t) \wedge g_1 \circ S_X(t) \wedge \cdots \wedge g_n \circ S_X(t)]\mathrm{d}t - \beta(\mu_0).$$

If $I_0 \in \mathcal{I}$ is an optimal solution to Problem (3.20), then $(I_1^*, \ldots, I_n^*) \in \mathcal{I}^n$ is an optimal solution to Problem (3.17), where, for $i = 1, \ldots, n$,

$$I_i^*(x) = \int_0^x \mathbb{I}_{\{A_i\}}(t) I_0'(t)\,\mathrm{d}t. \tag{3.21}$$

In the following, we shall consider the particular case when ρ is defined as Average Value-at-Risk (AVaR) and $n = 2$. Then Problem (3.17) becomes

$$\begin{aligned} &\min_{(I_1, I_2) \in \mathcal{I}^2} \mathrm{AVaR}_\alpha \left(X - I_1(X) - I_2(X) + P_{1,I_1} + P_{2,I_2}\right) &(3.22)\\ &\text{such that } P_{i,I_i} \triangleq \int_0^{\bar{X}} g_i \circ S_{I_i(X)}(t)\,\mathrm{d}t. \end{aligned}$$

Denote $T(I_1, I_2) \triangleq \mathrm{AVaR}_\alpha \left(X - I_1(X) - I_2(X)\right) + P_{1,I_1} + P_{2,I_2}$, for any $(I_1, I_2) \in \mathcal{I}^2$. Lemma 3.2.1 implies that $T(I_1, I_2)$ has the same minimal value as the following minimization problem

$$\begin{aligned} &\min_{I \in \mathcal{I}} \mathrm{AVaR}_\alpha(X - I(X) + P_I) &(3.23)\\ &\text{such that } P_I = \int_0^{\bar{X}} g \circ S_X(x)\,\mathrm{d}x, \end{aligned}$$

where $g(t) \triangleq g_1(t) \wedge g_2(t)$, for all $t \geq 0$.

Theorem 3.2.3 *The optimal pair of reinsurance contracts (I_1^*, I_2^*) for Problem (3.22) are*

$$I_i^*(x) = \int_0^x \mathbb{I}_{\{A_i \cap [0,d^*]\}}(t)\,\mathrm{d}t, \ \textit{for } i = 1, 2, \tag{3.24}$$

where d^* is any value such that $g \circ S_X(d^*) = S_X(d^*)/\alpha$. Moreover, within the set $\mathcal{I}^2$, the value function $T(I_1, I_2)$ achieves it minimal value

$$a + \frac{1}{\alpha}\int_{d^*}^{\bar{X}} S_X(t)\,\mathrm{d}t + \int_a^{d^*} g(S_X(t))\,\mathrm{d}t.$$

3.3 Appendix

3.3.1 Proofs

Proof of Lemma 3.1.1. In order to show the equivalence between (3.5) and (3.6), it is sufficient to show that $\rho(X - I(X) + P_I) = \sup_{\mu \in \mathcal{P}([0,1])} f(I, \mu)$, where f is defined by (3.7). By using Lemma 4.63 in [Follmer and Schied, 2004], for an arbitrary probability measure μ in $\mathcal{P}([0,1])$, there is a continuous increasing concave function $h_\mu : [0,1] \to [0,1]$ such that $h_\mu(0) = \mu(\{0\})$, $h_\mu(1) = 1$ and

$$h'_\mu(\alpha) = \int_{(\alpha,1]} \frac{1}{s}\, \mu(ds).$$

Taking into account the possibility of μ having non-zero measure at the single point set $\{0\}$, we have

$$\int_0^1 \mathrm{AVaR}_\alpha(X - I(X))\mu(\mathrm{d}\alpha) = R(\bar{X})\mu(\{0\}) + \int_{(0,1]} \mathrm{AVaR}_\alpha(X - I(X))\mu(\mathrm{d}\alpha).$$

By applying Fubini's Theorem, one gets, for any $I \in \mathcal{I}$,

$$\begin{aligned}
\int_{(0,1]} \mathrm{AVaR}_\alpha(X - I(X))\mu(\mathrm{d}\alpha) &= \int_{(0,1]} \int_0^\alpha \frac{1}{\alpha} \mathrm{VaR}_\xi(R(X)) d\xi\, \mu(\mathrm{d}\alpha) \\
&= \int_0^1 \int_{(\xi,1]} \frac{1}{\alpha} \mathrm{VaR}_\xi(R(X))\mu(d\alpha)\mathrm{d}\xi \\
&= \int_0^1 R(\mathrm{VaR}_\xi(X))h'(\xi)\mathrm{d}\xi \\
&= -\int_0^{\bar{X}} R(t)h'(S_X(t))\mathrm{d}S_X(t) \\
&= -\left(R(\bar{X})h(0) - R(0)h(1) - \int_0^{\bar{X}} h \circ S_X(t)R'(t)\mathrm{d}t\right) \\
&= -R(\bar{X})h(0) + \int_0^{\bar{X}} h_\mu(S_X(t))R'(t)\mathrm{d}t.
\end{aligned}$$

Therefore, we have

$$\int_0^1 \mathrm{AVaR}_\alpha(X - I(X))\mu(\mathrm{d}\alpha) = \int_0^{\bar{X}} h_\mu(S_X(t))R'(t)\mathrm{d}t.$$

It can be easily checked that $h_{a\mu+(1-a)\lambda} = ah_\mu + (1-a)h_\lambda$ for any μ and λ in $\mathcal{P}([0,1])$ and constant $a \in [0,1]$. It implies that ϕ_μ is linear with respect to μ. Thus, the function $f : \mathcal{I} \times \mathcal{P}([0,1]) \to \mathbb{R}$ defined by (3.7) can be written as follows,

$$f(I,\mu) = \int_0^{\bar{X}} h_\mu \circ S_X(t)\,(1 - I'(t))\,\mathrm{d}t + \int_0^{\bar{X}} g \circ S_X(t) I'(t)\mathrm{d}t - \beta(\mu).$$

Since ρ is translation invariant, we have

$$\begin{aligned} \rho\,(X - I(X) + P_I) &= \rho\,(X - I(X)) + P_I \\ &= \sup_{\mu \in \mathcal{P}([0,1])} \left(\int_0^1 \mathrm{AVaR}_\alpha(R(X))\mu(\mathrm{d}\alpha) - \beta(\mu) \right) + P_I \\ &= \sup_{\mu \in \mathcal{P}([0,1])} f(I,\mu). \end{aligned}$$

It implies that Problem (3.5) has the minimax expression (3.6). ■

Proof of Theorem 3.1.3. In order to apply the classical minimax theorem to interchange the minimum sign and the supremum sign in Problem (3.6), all conditions in Theorem 3.1.2 should be checked carefully:

1) Under the usual supremum norm, the family of all 1-Lipschitz functions on the compact interval $[0, \operatorname{ess\,sup} X]$ is a compact set.

2) $\mathcal{P}([0,1])$ is a convex set.

3) For each fixed $\mu \in \mathcal{P}([0,1])$, since $AVaR$ and VaR are both comonotonic additive, $f(\cdot,\mu)$ is convex on $\mathcal{I}$, indeed, $f(\lambda I_1 + (1-\lambda)I_2, \mu) = \lambda f(I_1,\mu) + (1-\lambda) f(I_2,\mu)$ for any I_1 and I_2 in $\mathcal{I}$ and $\lambda \in [0,1]$.

4) For each fixed $I \in \mathcal{I}$, $f(I,\mu)$ is concave on $\mathcal{P}([0,1])$ due to the convexity of function β.

Therefore, by applying the Minimax Theorem 3.1.2, one gets

$$\inf_{I \in \mathcal{I}} \sup_{\mu \in \mathcal{P}([0,1])} f(I,\mu) = \sup_{\mu \in \mathcal{P}([0,1])} \inf_{I \in \mathcal{I}} f(I,\mu),$$

which allows us to first solve the minimization problem $\inf\limits_{I \in \mathcal{I}} f(I,\mu)$ and find an expression for the minimizer I_μ, for an arbitrary fixed $\mu \in \mathcal{P}([0,1])$.

To this end, we are going to find the lower bound of function $f(\cdot,\mu)$ among $\mathcal{I}$. Since g, h, I' and $1-I'$ are all non-negative, one gets

$$\begin{aligned} f(I,\mu) &= \int_0^{\bar{X}} g\circ S_X(t)I'(t)\mathrm{d}t + \int_0^{\bar{X}} h_\mu\circ S_X(t)\left(1-I'(t)\right)\mathrm{d}t - \beta(\mu) \\ &\geq \int_0^{\bar{X}} \min\left\{g\circ S_X(t), h_\mu\circ S_X(t)\right\}\left(I'(t)+1-I'(t)\right)\mathrm{d}t - \beta(\mu) \\ &= \int_0^{\bar{X}} \phi_\mu\circ S_X(t)\mathrm{d}t - \beta(\mu). \end{aligned}$$

Conversely, it is easy to check that, the function I_μ define by (3.10) satisfies $I_\mu\in\mathcal{I}$ and

$$\begin{aligned} f(I_\mu,\mu) &= \int_{G_\mu} g\circ S_X(t)\mathrm{d}t + \int_{\mathbb{R}^+\setminus G_\mu} h_\mu\circ S_X(t)\mathrm{d}t - \beta(\mu) \\ &= \int_0^{\bar{X}} \phi_\mu\circ S_X(t)\mathrm{d}t - \beta(\mu). \end{aligned}$$

Thus, I_μ is a minimizer of $\inf\limits_{I\in\mathcal{I}} f(I,\mu)$ and, moreover, we have S is the saddle value of minimax problem (3.6).

Now, take a sequence of probability measures $\{\mu_n\}_{n=1}^\infty$ in $\mathcal{P}([0,1])$ such that

$$S = \lim_{n\to\infty} f(I_{\mu_n},\mu_n).$$

By Helly selection theorem, see [Billingsley, 1995] Page 336, there exists a subsequence of $\{\mu_n\}_{n=1}^\infty$ that weakly converges to a probability measure μ_0. Without loss of generality, assume $\mu_n\to\mu_0$ weakly as $n\to\infty$. Since $[0,1]$ is a closed interval in $\mathbb{R}$, one gets $\mu_n([0,1])\to\mu_0([0,1])$ and thus $\mu_0\in\mathcal{P}([0,1])$. By definition, for any $0<x<1$,

$$\begin{aligned} h_{\mu_n}(x) &= \mu_n(\{0\}) + \int_0^x \int_{(t,1]} \frac{1}{s}\,\mu_n(\mathrm{d}s)\mathrm{d}t \\ &= \mu_n(\{0\}) + \int_{(0,x]}\int_0^s \frac{1}{s}\,\mathrm{d}t\,\mu(\mathrm{d}s) + \int_{(x,1]}\int_0^x \frac{1}{s}\,\mathrm{d}t\,\mu(\mathrm{d}s) \\ &= \mu_n([0,x]) + x\int_{(x,1]} \frac{1}{s}\,\mu_n(\mathrm{d}s) \\ &= \int_0^1 \left[\mathbb{I}_{\{[0,x]\}}(s) + \frac{x}{s}\,\mathbb{I}_{\{(x,1]\}}(s)\right]\mu_n(\mathrm{d}s). \end{aligned}$$

Note that, for any $0 < x \le 1$, the integrand on the right hand side of the last equality is a continuous function on $[0,1]$. It implies that, for any $0 < x \le 1$, as $n \to \infty$, we have

$$\begin{aligned} h_{\mu_n}(x) &= \int_0^1 \left[\mathbb{I}_{\{[0,x]\}}(s) + \frac{x}{s}\, \mathbb{I}_{\{(x,1]\}}(s) \right] \mu_n(\mathrm{d}s) \\ &\to \int_0^1 \left[\mathbb{I}_{\{[0,x]\}}(s) + \frac{x}{s}\, \mathbb{I}_{\{(x,1]\}}(s) \right] \mu_0(\mathrm{d}s) = h_{\mu_0}(x). \end{aligned}$$

It should be pointed that $h_{\mu_n}(0)$ may not converge to $h_{\mu_0}(0)$. However, the Borel set $\{0\}$ has Lebesgue measure zero, so the discontinuity of sequence $h_{\mu_n}(0)$ won't effect the Lebesgue measure. Moreover, we have

$$\lim_{n\to\infty} \int_0^{\bar{X}} \min\left\{ h_{\mu_n} \circ S_X(t), g \circ S_X(t) \right\} \mathrm{d}t = \int_0^{\bar{X}} \min\left\{ h_{\mu_0} \circ S_X(t), g \circ S_X(t) \right\} \mathrm{d}t.$$

Meanwhile, the fact that β is non-negative lower-semi-continous function implies that

$$\limsup_{n\to\infty} -\beta(\mu_n) = -\liminf_{n\to\infty} \beta(\mu_n) \le -\beta(\mu_0).$$

Thus,

$$\begin{aligned} S = \lim_{n\to\infty} f(I_{\mu_n}, \mu_n) &\le \limsup_{n\to\infty} \int_0^{\bar{X}} \phi_\mu \circ S_X(t)\mathrm{d}t + \limsup_{n\to\infty} -\beta(\mu_n) \\ &\le \int_0^{\bar{X}} \min\left\{ h_{\mu_0} \circ S_X(t), g \circ S_X(t) \right\} \mathrm{d}t - \beta(\mu_0) \\ &= f(I_{\mu_0}, \mu_0) \le S. \end{aligned}$$

As a consequence, we can conclude that the minimax problem (3.6) has saddle-value:

$$\begin{aligned} \min_{I\in\mathcal{I}} \sup_{\mu\in\mathcal{P}([0,1])} f(I,\mu) &= \sup_{\mu\in\mathcal{P}([0,1])} \min_{I\in\mathcal{I}} f(I,\mu) \\ &= \sup_{\mu\in\mathcal{P}([0,1])} f(I_\mu, \mu) \\ &= \sup_{\mu\in\mathcal{P}([0,1])} \left\{ \int_0^{\bar{X}} \phi_\mu \circ S_X(t)\mathrm{d}t - \beta(\mu) \right\} \\ &= \int_0^{\bar{X}} \phi_{\mu_0} \circ S_X(t) dt - \beta(\mu_0). \end{aligned}$$

■

Proof of Theorem 3.1.4. From Theorem 3.1.3, we get

$$\inf_{I\in\mathcal{I}} \rho\left(X - I(X) + P_I\right) = S.$$

Suppose a sequence $\{I_n\}_{n=1}^{\infty} \subset \mathcal{I}$ satisfies

$$\lim_{n\to\infty} \rho\left(X - I_n(X) + P_{I_n}\right) = \inf_{I\in\mathcal{I}} \rho\left(X - I(X) + P_I\right) = S.$$

Under the usual supremum norm, the family of all 1-Lipschitz functions on the compact interval $[0, \bar{X}]$ is a compact set where $\bar{X} = \operatorname{ess\,sup} X < \infty$. Thus, there is a subsequence of $\{I_n\}_{n=1}^{\infty}$ that converges to a 1-Lipschitz function I^* with respect to the supremum norm. Without loss of generality, take $\lim_{n\to\infty} \|I_n - I^*\|_\infty = 0$. Since $(\Omega, \mathcal{F}, \mathbb{P})$ is an atomless probability space, it is easy to verify that $I_n(X) \xrightarrow{\mathbf{P}} I^*(X)$, and thus $X - I_n(X) \xrightarrow{\mathbf{P}} X - I^*(X)$. Due to the Fatou property of ρ, which is automatically satisfied by every law invariant convex risk measure, we have

$$\rho(X - I^*(X)) \le \liminf_{n\to\infty} \rho(X - I_n(X)).$$

Meanwhile, $I_n(X) \xrightarrow{\mathbf{P}} I^*(X)$ implies $I_n(X) \xrightarrow{d} I^*(X)$, and thus, for any $t \ge 0$,

$$\lim_{n\to\infty} g \circ S_{I_n(X)}(t) = g \circ S_{I^*(X)}(t).$$

Since $I(x) \le x$ for any $x \ge 0$, the survival distribution $S_{I_n(X)}(t)$ is bounded above by $S_X(t)$, and then $g \circ S_{I_n(X)}(t) \le g \circ S_X(t)$ for any $t \ge 0$ because g is non-decreasing. Obviously, $g \circ S_X(t)$ is integrable in the sense that $\int_0^\infty g \circ S_X(t)\mathrm{d}t < \infty$. By the Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} P_{I_n} = \lim_{n\to\infty} \int_0^\infty g \circ S_{I_n(X)}(t)\underset{\cdot}{\mathrm{t}} = \int_0^\infty g \circ S_{I^*(X)}(t)\mathrm{d}t = P_{I^*}.$$

Therefore,

$$\begin{aligned}
\rho(X - I^*(X) + P_{I^*}) &= \rho(X - I^*(X)) + P_{I^*} \\
&\le \liminf_{n\to\infty} \rho(X - I_n(X)) + \lim_{n\to\infty} P_{I_n} = \liminf_{n\to\infty} \rho(X - I_n(X) + P_{I_n}) \\
&\le S,
\end{aligned}$$

and then $\rho(X - I^*(X) + P_{I^*}) = S$, where $I^* \in \mathcal{I}$. That is, $\rho(X - I^*(X) + P_{I^*}) = \min_{I\in\mathcal{I}} \rho\left(X - I(X) + P_I\right) = S$, namely, S given by (3.11) is the minimal value of Problem (3.3) exists and is achieved by I^*. ■

Proof of Proposition 3.1.5. Suppose I_0 is one optimal solution of Problem 3.3, by Theorem (3.1.3) we have

$$\rho\left(X - I_0(X) + P_{I_0}\right) = \sup_{\mu \in \mathcal{P}([0,1])} f(I_0, \mu) = S.$$

It follows that,

$$S = f(I_{\mu_0}, \mu_0) = \min_{I \in \mathcal{I}} f(I, \mu_0) \leq f(I_0, \mu_0) \leq \sup_{\mu \in \mathcal{P}([0,1])} f(I_0, \mu) = S,$$

and thus $f(I_{\mu_0}, \mu_0) = f(I_0, \mu_0)$. By definition (3.10),

$$f\left(I_{\mu_0}, \mu_0\right) = \int_{G_{\mu_0}} g \circ S_X(t) \mathrm{d}t + \int_{\mathbb{R}_+ / G_{\mu_0}} h_{\mu_0} \circ S_X(t) \mathrm{d}t.$$

A direct calculation gives us

$$\begin{aligned}
0 = f(I_0, \mu_0) - f(I_{\mu_0}, \mu_0) &= \int_0^{\bar{X}} g \circ S_X(t) I_0'(t) \mathrm{d}t + \int_0^{\bar{X}} h_{\mu_0} \circ S_X(t)(1 - I_0'(t)) \mathrm{d}t \\
&\quad - \int_{G_{\mu_0}} g \circ S_X(t) \mathrm{d}t - \int_{\mathbb{R}_+ / G_{\mu_0}} h_{\mu_0} \circ S_X(t) \mathrm{d}t \\
&= \int_{G_{\mu_0}} \left[h_{\mu_0} \circ S_X(t) - g \circ S_X(t)\right] (1 - I_0'(t)) \mathrm{d}t \\
&\quad + \int_{\mathbb{R}_+ / (G_{\mu_0} \cup E_{\mu_0})} \left(g \circ S_X(t) - h_{\mu_0} \circ S_X(t)\right) I_0'(t) \mathrm{d}t \\
&\quad + \int_{E_{\mu_0}} \left[g \circ S_X(t) I_0'(t) + h_{\mu_0} \circ S_X(t)(1 - I_0'(t)) - h_{\mu_0} \circ S_X(t)\right] \mathrm{d}t.
\end{aligned}$$

Note that, the first two terms on the right-hand side of the second equality are both non-negative and the third term is zero. Therefore, we must have $I_0'(t) = 0$ for any $t \in \mathbb{R}^+ / (G_{\mu_0} \cup E_{\mu_0})$ and $I_0'(t) = 1$ for any $t \in G_{\mu_0}$. Since I satisfies "slow growing" property, on set E_μ, its first derivative I' is equal to some function between $[0, 1]$ and denoted by $\alpha(t)$. ■

Proof of Theorem 3.1.6. The existence of μ is given by [Kusuoka, 2001]. In his paper, Kusuoka showed that an equivalent expression for a law invariant and comonotone coherent risk measure with the Fatou property ρ is

$$\rho(X) = \int_{[0,1]} \mathrm{AVaR}_\alpha(X) \mu(\mathrm{d}\alpha), \text{ for any } X \in \mathcal{L}^\infty,$$

where μ is a probability measure on $[0, 1]$. Comparing with the expression (3.4), under the additional assumption that ρ is comonotone, we have $\beta \equiv 0$ and μ is the maximizer for any $X \in \mathcal{L}^\infty$. It is easy to see that μ_0 in Theorem 3.1.3 turns out to be μ. Therefore, for any $I \in \mathcal{I}$, one gets

$$\begin{aligned}
\rho(X - I(X)) + P_I &= \int_0^{\bar{X}} h_\mu \circ S_X(x)\,[1 - I'(x)]\,\mathrm{d}x + \int_0^{\bar{X}} g \circ S_X(x) I'(x)\mathrm{d}x - \beta(\mu) \\
&\geq \int_0^{\bar{X}} \min\{h_\mu \circ S_X(x), g \circ S_X(x)\}\,\mathrm{d}x - \beta(\mu) \\
&= \rho(X - I_0(X)) + P_{I_0},
\end{aligned}$$

and it implies that I_0 is an optimal reinsurance contract. ■

Proof of Lemma 3.2.1. For any $(I_1, \ldots, I_n) \in \mathcal{I}^n$, it is easy to check that the total ceded loss $I \triangleq \sum_{i=1}^n I_i$ can be served a feasible reinsurance contract in the single reinsurer case, namely, $I \in \mathcal{I}$. The total premium from n reinsurers is

$$\begin{aligned}
\sum_{i=1}^n P_{i,I_i} &= \sum_{i=1}^n \int_0^{\bar{X}} g_i \circ S_X(t) I_i'(t)\mathrm{d}t \\
&\geq \int_0^{\bar{X}} \left[(g_1 \circ S_X(t) \wedge \cdots \wedge g_n \circ S_X(t)) \sum_{i=1}^n I_i'(t) \right] \mathrm{d}t \\
&= \int_0^{\bar{X}} g \circ S_X(t) I'(t)\mathrm{d}t \\
&= P_I,
\end{aligned}$$

where the inequality holds because g_i and I_i, $i = 1, \ldots, n$ are all non-negative. Thus, for any given $(I_1, \ldots, I_n) \in \mathcal{I}^n$, the total ceded loss $I \in \mathcal{I}$ and moreover $\sum_{i=1}^n P_{i,I_i} \geq P_I$, where P_I is defined by (3.18). This implies that

$$\min_{(I_1,\ldots,I_n)\in\mathcal{I}^n} \rho\left(X - \sum_{i=1}^n I_i(X) + \sum_{i=1}^n P_{i,I_i}\right) \geq \min_{I\in\mathcal{I}} \rho\left(X - I(X) + P_I\right). \tag{3.25}$$

Conversely, any $I \in \mathcal{I}$ can be decomposed into the sum of n reinsurance contracts. Indeed, $I = \sum_{i=1}^n I_i$ if $I_i(0) = 0$, $i = 1, \ldots, n$ and the derivative of I_i, $i = 1, \ldots, n$ are $I_i' = \mathbb{I}_{\{A_1\}} I'$, where

$$\begin{aligned}
&A_i \triangleq \{t \geq 0 : g_i \circ S_X(t) < g_j \circ S_X(t),\ j = 1, \ldots, n,\ j \neq i\}, \text{ for } i = 1, \ldots, n-1; \\
&A_n \triangleq \{t \geq 0 : g_n \circ S_X(t) \geq g_j \circ S_X(t),\ j = 1, \ldots, n-1\}.
\end{aligned}$$

It is easy to check that the functions I_i, $i = 1, \ldots, n$ are non-negative, Lipschitz-continuous and non-decreasing which imply that $(I_1, \ldots, I_n) \in \mathcal{I}^n$. Furthermore, from $\sum_{i=1}^n I_i = I$, one gets

$$\rho\left(X - \sum_{i=1}^n I_i(X)\right) = \rho\left(X - I(X)\right);$$

and

$$P_I = \int_0^{\bar{X}} \left[g_1 \circ S_X(t) \wedge \cdots \wedge g_n \circ S_X(t)\right] I'(t)\mathrm{d}t = \sum_{i=1}^n \int_{A_i} g_i \circ S_X(t) I'(t)\mathrm{d}t = \sum_{i=1}^n P_{i,I_i}.$$

Therefore, the inequality in expression (3.25) is actually an equality and this gives us the result. ■

Proof of Proposition 3.2.2. The minimal value of Problem (3.17) is given by Lemma 3.2.1 and Theorem 3.1.3. Secondly, consider the n real-valued functions defined by (3.21), it is easy to see $(I_1^*, \ldots, I_n^*) \in \mathcal{I}^n$ and $\sum_{i=1}^n I_i^* = I_0$. It implies that,

$$\rho\left(X - \sum_{i=1}^n I_i^*(X)\right) = \rho\left(X - I_0(X)\right), \quad \text{and} \quad \sum_{i=1}^n P_{i,I_i^*} = P_{I_0},$$

and moreover,

$$\begin{aligned}\rho\left(X - \sum_{i=1}^n I_i^*(X)\right) + \sum_{i=1}^n P_{i,I_i^*} &= \rho\left(X - I_0(X)\right) + P_{I_0} \\ &= \int_0^{\bar{X}} \left[h_{\mu_0} \circ S_X(t) \wedge g \circ S_X(t)\right] \mathrm{d}t - \beta\left(\mu_0\right) \\ &= \int_0^{\bar{X}} \left[h_{\mu_0} \circ S_X(t) \wedge g_1 \circ S_X(t) \wedge \cdots \wedge g_n \circ S_X(t)\right] \mathrm{d}t - \beta(\mu_0).\end{aligned}$$

Therefore, $(I_1^*, \ldots, I_n^*)$ is the minimizer of Problem (3.17). ■

Proof of Theorem 3.2.3. Denote $g(t) \triangleq g_1(t) \wedge g_2(t)$, for all $t \geq 0$. Lemma 3.2.1 implies that

$$\min_{(I_1,I_2)\in\mathcal{I}^2} T\left(I_1, I_2\right) = \min_{I\in\mathcal{I}} \mathrm{AVaR}_\alpha\left(X - I(X) + P_I\right), \tag{3.26}$$

where $P_I = \int_0^{\bar{X}} g \circ S_X(x)\,\mathrm{d}x$. From (3.16) in Example 3.1.1, the optimal solution to the minimization problem on the right hand side of the equation (3.26) has form $I^*(x) =$

$x - (x - d^*)^+$, where $d^* \geq a = \mathrm{VaR}_\alpha(X)$ satisfies the equation $S_X(d^*)/\alpha = g \circ S_X(d^*)$ if $g'(0) > \frac{1}{\alpha}$, and $d^* = \bar{X}$ if $g'(0) \leq \frac{1}{\alpha}$. Therefore,

$$\min_{I \in \mathcal{I}} \mathrm{AVaR}_\alpha \left(X - I(X) + P_I\right) = \mathrm{AVaR}_\alpha(X - I^*(X) + P_{I^*}). \tag{3.27}$$

Use (3.19) to define two sets A_1 and A_2 as follows

$$A_1 = \left\{t > 0 : g_1 \circ S_X(t) < g_2 \circ S_X(t)\right\}, \text{ and } A_2 = \mathbb{R}^+ \setminus A_1.$$

By Proposition 3.2.2, the optimal pair $(I_1^*, I_2^*) \in \mathcal{I}^2$ to Problem (3.22) is

$$I_i^*(x) = \int_0^x \mathbb{I}_{\{A_i\}}(t)(I^*)'(t)\,\mathrm{d}t = \int_0^x \mathbb{I}_{\{A_i \cap [0,d^*]\}}(t)\,\mathrm{d}t, \text{ for } i = 1, 2.$$

Moreover, it is easy to check that

$$T(I_1^*, I_2^*) = a + \frac{1}{\alpha}\int_{d^*}^{\bar{X}} S_X(t)\,\mathrm{d}t + \int_a^{d^*} g \circ S_X(t)\mathrm{d}t.$$

■

Alternative proof for Theorem 3.1.7

Theorem 3.1.7 is saying that

$$\rho(X - I_{\mu_0}(X) + P_{\mu_0}) = \max_{\mu \in \mathcal{P}([0,1])} f(I_{\mu_0}, \mu) = f(I_{\mu_0}, \mu_0) = S.$$

where μ_0 is the minimizer of $f(I_\mu, \mu)$. Therefore, the curial part is to show that the concave function $f(I_{\mu_0}, \mu)$, with respect to μ and achieves its maximal value at point μ_0.

Fix a point $\mu \in \mathcal{P}([0,1])$, define its corresponding functions

$$F_{I_\mu}(\lambda) \triangleq f(I_\mu, \lambda), \forall \lambda \in \mathcal{P}([0,1]),$$

and define

$$F(\lambda) \triangleq f(I_\lambda, \lambda), \forall \lambda \in \mathcal{P}([0,1]).$$

Then μ_0 is the minimizer of $F(\mu)$. For a concave function defined on the real line, as long as it has zero first derivative at some point, it achieves maximal value at this point. However, both functions $F_{I_\mu}(\lambda)$ and $F(\lambda)$ are functional on the probability space $\mathcal{P}([0,1])$ and the classical definition of derivative is not valid in this case. To overcome this difficulty, we need to introduce a more general definition of derivative.

For the fixed point $\mu \in \mathcal{P}([0,1])$, define the *directional derivative* of functions $F_{I_{\mu_0}}$ and F at point μ along direction $\lambda \in \mathcal{P}([0,1])$ as follows:

$$\begin{aligned} F'_{I_\mu}(\mu)[\lambda] &\triangleq \lim_{a\uparrow 1^-} \frac{F_{I_\mu}(a\mu+(1-a)\lambda)-F_{I_\mu}(\mu)}{1-a}, \\ F'(\mu)[\lambda] &\triangleq \lim_{a\uparrow 1^-} \frac{F(a\mu+(1-a)\lambda)-F(\mu)}{1-a}. \end{aligned}$$

Each probability measure $\lambda \in \mathcal{P}([0,1])$ represents a valid direction.

Both $F'_{I_\mu}(\mu)[\lambda]$ and $F'(\mu)[\lambda]$ are well-defined. Indeed, for fixed μ and λ, it can be shown by the following argument that $\left[F(a\mu+(1-a)\lambda)-F(\mu)\right]/(1-a)$ is non-decreasing with respect to a: suppose $0 \le a < b \le 1$, then

$$b\mu+(1-b)\lambda = c\mu + (1-c)\left(a\mu+(1-a)\lambda\right),$$

where $c = 1-\frac{1-b}{1-a} \in (0,1)$. It implies that, together with the concavity property of function F,

$$\begin{aligned} F\left(b\mu+(1-b)\lambda\right)-F(\mu) &\ge (1-c)\left(F\left(a\mu+(1-a)\lambda\right)-F(\mu)\right) \\ &= (1-b)\frac{F\left(a\mu+(1-a)\lambda\right)-F(\mu)}{1-a}. \end{aligned}$$

First, we are going to show the following two lemmas.

Lemma 3.3.1 *For the same $\mu_0 \in \mathcal{P}([0,1])$ as in Theorem 3.1.7, we have that*

$$F'_{I_{\mu_0}}(\mu_0)[\lambda] = F'(\mu_0)[\lambda], \textit{ for any } \lambda \in \mathcal{P}([0,1]).$$

Proof. Fix μ and λ in $\mathcal{P}([0,1])$, denote $\gamma_a = a\mu+(1-a)\lambda$ for $a \in [0,1]$. From the definition, one gets

$$\begin{aligned} h'_{\gamma_a}(x) &= \int_{(x,1]} \frac{1}{s}\,\gamma_a(\mathrm{d}s) = \int_{(x,1]} \frac{1}{s}\left[a\mu(\mathrm{d}s)+(1-a)\lambda(\mathrm{d}s)\right] \\ &= a\int_{(x,1]} \frac{1}{s}\,\mu(\mathrm{d}s) + (1-a)\int_{(x,1]} \frac{1}{s}\,\lambda(\mathrm{d}s) = ah'_\mu(x)+(1-a)h'_\lambda(x), \end{aligned}$$

and thus $h_{\gamma_a}(x) = ah_\mu(x) + (1-a)h_\lambda(x)$. Therefore,

$$F_{I_\mu}(\gamma_a) - F_{I_\mu}(\mu) = f(I_\mu, \gamma_a) - f(I_\mu, \mu) = \int_{G_\mu^c} h_{\gamma_a} \circ S_X(t) - h_\mu \circ S_X(t)\mathrm{d}t - [\beta(\gamma_a) - \beta(\mu)]$$
$$= (1-a)\int_{G_\mu^c} [h_\lambda \circ S_X(t) - h_\mu \circ S_X(t)]\, \mathrm{d}t - [\beta(\gamma_a) - \beta(\mu)]\,,$$

where $G_\mu^c = \mathbb{R}/G_\mu$. It implies that

$$F'_{I_\mu}(\mu)[\lambda] = \lim_{a\uparrow 1^-} \frac{F_{I_\mu}(\gamma_a) - F_{I_\mu}(\mu)}{1-a} = \int_{G_\mu^c} [h_\lambda \circ S_X(t) - h_\mu \circ S_X(t))]\, \mathrm{d}t - \beta'(\mu)[\lambda].$$

In what follows, we calculate the directional derivative of the function F. Note that

$$\begin{aligned}
& F(\gamma_a) - F(\mu) = f(I_{\gamma_a}, \gamma_a) - f(I_\mu, \mu) \\
= \ & \int_{G_{\gamma_a}} g \circ S_X(t))\mathrm{d}t + \int_{G_\gamma^c} h_{\gamma_a} \circ S_X(t))\mathrm{d}t \\
& - \left(\int_{G_\mu} \left[g \circ S_X(t))\mathrm{d}t + \int_{G_\mu^c} h_\mu \circ S_X(t)) \right] \mathrm{d}t \right) - [\beta(\gamma_a) - \beta(\mu)] \\
= \ & \int_{G_{\gamma_a} \cap G_\mu^c} g \circ S_X(t) - h_\mu \circ S_X(t)\mathrm{d}t + \int_{G_{\gamma_a}^c \cap G_\mu} [h_{\gamma_a} \circ S_X(t) - g \circ S_X(t))]\, \mathrm{d}t \\
& + \int_{G_{\gamma_a}^c \cap G_\mu^c} [h_{\gamma_a} \circ S_X(t) - h_\mu \circ S_X(t)]\, \mathrm{d}t - (\beta(\gamma_a) - \beta(\mu)) \\
= \ & \int_{G_{\gamma_a} \cap G_\mu^c} [g \circ S_X(t) - h_\mu \circ S_X(t)]\, \mathrm{d}t + \int_{G_{\gamma_a}^c \cap G_\mu} [h_{\gamma_a} \circ S_X(t) - g \circ S_X(t)]\, \mathrm{d}t \\
& + F_{I_\mu}(\gamma_a) - F_{I_\mu}(\mu) - \int_{G_{\gamma_a} \cap G_\mu^c} [h_{\gamma_a} \circ S_X(t) - h_\mu \circ S_X(t))]\, \mathrm{d}t \\
= \ & \int_{G_{\gamma_a} \cap G_\mu^c} [g \circ S_X(t) - h_{\gamma_a} \circ S_X(t)]\, \mathrm{d}t + \int_{G_{\gamma_a}^c \cap G_\mu} [h_{\gamma_a} \circ S_X(t) - g \circ S_X(t)]\, \mathrm{d}t \\
& + F_{I_\mu}(\gamma_a) - F_{I_\mu}(\mu).
\end{aligned}$$

It implies that

$$\begin{aligned}
F'(\mu)[\lambda] - F'_{I_\mu}(\mu)[\lambda] \ = \ & \lim_{a\uparrow 1^-} \frac{1}{1-a} \int_{G_{\gamma_a} \cap G_\mu^c} [g \circ S_X(t) - h_{\gamma_a} \circ S_X(t)]\, \mathrm{d}t \qquad (3.28) \\
& + \lim_{a\uparrow 1^-} \frac{1}{1-a} \int_{G_{\gamma_a}^c \cap G_\mu} [h_{\gamma_a} \circ S_X(t) - g \circ S_X(t)]\, \mathrm{d}t,
\end{aligned}$$

and thus, as long as the right hand side of equation (3.28) equals zero, the expected result holds. To this end, we are going to check these two terms separately.

1) On the set $G_{\gamma_a} \cap G_{\mu}^c$, we have

$$h_\mu(S_X(t)) \le g \circ S_X(t) < h_{\gamma_a} \circ S_X(t) = a h_\mu \circ S_X(t) + (1-a) h_\lambda \circ S_X(t);$$

therefore,

$$(1-a) \int\limits_{G_{\gamma_a} \cap G_{\mu}^c} [g \circ S_X(t) - h_\lambda \circ S_X(t)] \, \mathrm{d}t \le \int\limits_{G_{\gamma_a} \cap G_{\mu}^c} [g \circ S_X(t) - h_{\gamma_a} \circ S_X(t)] \, \mathrm{d}t < 0.$$

Note that

$$G_{\gamma_a} \cap G_{\mu}^c = \left\{ t \ge 0 : 0 \le g(S_X(t)) - h_\mu(S_X(t)) < \frac{1-\alpha}{\alpha} \left(h_\lambda(S_X(t)) - g(S_X(t)) \right) \right\},$$

thus,

$$A \triangleq \bigcap_{a \uparrow 1} \left(G_{\gamma_a} \cap G_{\mu}^c \right) = \{ t \ge 0 : g\left(S_X(t)\right) = h_\mu\left(S_X(t)\right) < h_\lambda\left(S_X(t)\right) \} \subset E_\mu,$$

and

$$\int\limits_{A} [g \circ S_X(t) - h_\lambda \circ S_X(t)] \, \mathrm{d}t \le \lim_{a \uparrow 1^-} \frac{1}{1-a} \int\limits_{G_{\gamma_a} \cap G_{\mu}^c} [g \circ S_X(t) - h_{\gamma_a} \circ S_X(t)] \, \mathrm{d}t \le 0.$$

In particular, for μ_0, under the assumption that $g = h_{\mu_0}$ holds only on a Lebesgue's measure zero set, we have

$$\lim_{a \uparrow 1^-} \frac{1}{1-a} \int\limits_{G_{\gamma_a} \cap G_{\mu_0}^c} g \circ S_X(t) - h_{\gamma_a} \circ S_X(t) dt = 0.$$

2) On the set $G_{\gamma_a}^c \cap G_\mu$, we have

$$h_{\gamma_a} \circ S_X(t) = a h_\mu \circ S_X(t) + (1-a) h_\lambda \circ S_X(t) \le g \circ S_X(t) < h_\mu \circ S_X(t),$$

therefore

$$(1-a) \int\limits_{G_{\gamma_a}^c \cap G_\mu} [h_\lambda \circ S_X(t) - g \circ S_X(t)] \, \mathrm{d}t \le \int\limits_{G_{\gamma_a}^c \cap G_\mu} [h_{\gamma_a} \circ S_X(t) - g \circ S_X(t)] \, \mathrm{d}t < 0.$$

Denote $M \triangleq \inf_{t \in G^c_{\gamma_a} \cap G_\mu} \{h_\lambda \circ S_X(t) - g \circ S_X(t)\}$, then $M > -\infty$ and

$$\begin{aligned} G^c_{\gamma_a} \cap G_\mu &= \left\{ t \geq 0 : \frac{1-a}{a} \left[h_\lambda \circ S_X(t) - g \circ S_X(t)\right] \leq g \circ S_X(t) - h_\mu \circ S_X(t) < 0 \right\} \\ &\subseteq \left\{ t \geq 0 : \frac{1-a}{a} M \leq g \circ S_X(t) - h_\mu \circ S_X(t) < 0 \right\}. \end{aligned}$$

It implies that the Lebesgue measure of $G^c_{\gamma_a} \cap G_\mu$ converges to zero as $a \uparrow 1$ for any $\mu \in \mathcal{P}([0,1])$. Thus,

$$\lim_{a \uparrow 1^-} \frac{1}{1-a} \int\limits_{G^c_{\gamma_a} \cap G_{\mu_0}} \left[h_{\gamma_a} \circ S_X(t) - g \circ S_X(t)\right] \mathrm{d}t = 0.$$

Therefore, Equation (3.28) implies that for any $\lambda \in \mathcal{P}([0,1])$, we have

$$F'(\mu_0)[\lambda] = F'_{I_{\mu_0}}(\mu_0)[\lambda]$$

as required. ■

Note that $\mathcal{P}([0,1])$ is a convex subset of the set of all signed measures on $[0,1]$, denoted by $\mathcal{M}([0,1])$, which is a Banach space. Denote $\mathcal{M}^*$ to be the dual space of $\mathcal{M}([0,1])$, i.e. $\mathcal{M}^*$ is the set of all linear functionals on $\mathcal{P}([0,1])$. For any function $H : \mathcal{P}([0,1]) \to \mathbb{R}$, if H takes a finite value at $\mu \in \mathcal{P}([0,1])$, define

$$\partial H(\mu) \triangleq \left\{ \mu^* \in \mathcal{M}^* : H(\lambda) \leq H(\mu) + \langle \mu^*, \lambda \rangle - \langle \mu^*, \mu \rangle, \text{ for any } \lambda \in \mathcal{M} \right\},$$

where $\langle \mu^*, \lambda \rangle$ is the value of linear functional $\mu^* \in \mathcal{M}^*$ at probability measure λ.

For $\mu_0 \in \mathcal{P}([0,1])$, define a corresponding function $\psi : \mathcal{P}([0,1]) \to \mathbb{R}$ via $\psi(\lambda) = F'(\mu_0)[\lambda]$ for any $\lambda \in \mathcal{P}([0,1])$. Since μ_0 gives the maximal value of $F(\mu)$, function ψ is always non-positive and thus finite.

Lemma 3.3.2 *Under the same condition as in Theorem 3.1.7, the sets $\partial F(\mu_0)$ and $\partial \psi(\mu_0)$ are non-empty and*

$$\partial F(\mu_0) = \partial \psi(\mu_0).$$

Proof. It is known that $F(\mu)$ achieves its maximal value at probability measure μ_0, i.e. $F(\lambda) \leq F(\mu_0)$ for any $\lambda \in \mathcal{P}([0,1])$. It implies that $0 \in \partial F(\mu_0)$ and thus $\partial F(\mu_0) \neq \emptyset$. For any $\mu^* \in \partial F(\mu_0)$, i.e. μ^* such that

$$F(\lambda) \leq F(\mu_0) + \langle \mu^*, \lambda \rangle - \langle \mu^*, \mu_0 \rangle, \text{ for any } \lambda \in \mathcal{P}([0,1]),$$

one gets

$$\begin{aligned} \psi(\lambda) &= \lim_{a \uparrow 1^-} \frac{F(a\mu_0 + (1-a)\lambda) - F(\mu_0)}{1-a} \\ &\leq \lim_{a \uparrow 1^-} \frac{\langle \mu^*, a\mu_0 + (1-a)\lambda) \rangle - \langle \mu^*, \mu_0 \rangle}{1-a} \\ &= \langle \mu^*, \lambda \rangle - \langle \mu^*, \mu_0 \rangle. \end{aligned}$$

It implies that $\mu^* \in \partial\psi(\mu_0)$ because $\psi(\mu_0) = 0$ and moreover, $\partial\psi(\mu) \neq \emptyset$.

Conversely, for any $\mu^* \in \partial\psi(\mu_0)$,

$$\psi(\lambda) \leq \psi(\mu_0) + \langle \mu^*, \lambda \rangle - \langle \mu^*, \mu_0 \rangle = \langle \mu^*, \lambda \rangle - \langle \mu^*, \mu_0 \rangle,$$

holds for any $\lambda \in \mathcal{P}([0,1])$. Since

$$F'(\mu_0)[\lambda] = \sup_{(1-a)>0} \frac{F(a\mu + (1-a)\lambda) - F(\mu_0)}{1-a},$$

one gets

$$F(\lambda) - F(\mu_0) \leq F'(\mu_0)[\lambda] = \psi(\lambda) \leq \langle \mu^*, \lambda \rangle - \langle \mu^*, \mu_0 \rangle.$$

It implies that $\mu^* \in \partial F(\mu_0)$. ■

As a consequence of Lemma 3.3.1, an equivalent definition for ψ is $\psi(\lambda) = F'_{I_{\mu_0}}(\mu_0)[\lambda]$ for any $\lambda \in \mathcal{P}([0,1])$. Then, by using the same argument as in Lemma 3.3.2, we have

$$\partial F(\mu_0) = \partial\psi(\mu_0) = \partial F_{I_{\mu_0}}(\mu_0).$$

Therefore, $0 \in \partial F_{I_{\mu_0}}(\mu_0)$, or equivalently, function $F_{I_{\mu_0}}$ achieves its maximal value among the set $\mathcal{P}([0,1])$ at μ_0.

Now, we are ready to prove Theorem 3.1.7.

Proof of Theorem 3.1.7. From the above argument, one gets

$$\rho\left(X - I_{\mu_0}(X)\right) + P_{I_{\mu_0}} = \sup_{\mu \in \mathcal{P}([0,1])} f(I_{\mu_0}, \mu) = \sup_{\mu \in \mathcal{P}([0,1])} F_{I_{\mu_0}}(\mu) = F_{I_{\mu_0}}(\mu_0).$$

Theorem 3.1.3 shows that the minimal value for Problem (3.3) is $f(I_{\mu_0}, \mu_0) = F_{I_{\mu_0}}(\mu_0)$, thus $\rho(X - I(X)) + P_I$ achieve its minimal value at I_{μ_0}, i.e.

$$\min_{I \in \mathcal{I}} \rho(X - I(X) + P_I) = \rho(X - I_{\mu_0}(X)) + P_{I_{\mu_0}}.$$

Indeed, (I_{μ_0}, μ_0) is the saddle point of the minimax function $f(I, \mu)$ on $\mathcal{I} \times \mathcal{P}([0,1])$, i.e.

$$\sup_{\mu \in \mathcal{P}([0,1])} f(I_{\mu_0}, \mu) = F_{I_{\mu_0}}(\mu_0) = f(I_{\mu_0}, \mu_0) = \min_{I \in \mathcal{I}} f(I, \mu_0).$$

■

Chapter 4

Joint perspectives of both an insurer and a reinsurer

In this chapter, we study optimal reinsurance designs from the perspectives of both an insurer and a reinsurer and take into account both an insurer's aims and a reinsurer's goals in reinsurance contract designs. One of the main objectives for an insurer when buying a reinsurance is to control his risk, while one of the main goals for a reinsurer when selling a reinsurance is to make a profit. Of course, a reinsurer also worries about his own risk when selling a reinsurance contract and needs to control his risk as well.

We assume both the insurer and the reinsurer use VaR to measure their own losses and develop optimal reinsurance contracts that minimize the convex combination of the VaR risk measures of the insurer's loss and the reinsurer's loss under two types of constraints. The constraints describe the interests of both the insurer and the reinsurer. With the first type of constraints, the insurer and the reinsurer have their own limit on the VaR of their own loss. With the second type of constraints, the insurer has a limit on the VaR of his loss while the reinsurer has a target on his profit in selling a reinsurance contract. For both types of constraints, we derive the optimal reinsurance forms within a wide class of reinsurance policies and under the expected value reinsurance premium principle. These optimal reinsurance forms are more complicated than the optimal reinsurance contracts from the perspective of one party only. The proposed models can also be reduced to the problems of minimizing the VaR of one party's loss under the constraints on the interests of both the insurer and the reinsurer.

In this chapter, we assume the underlying non-negative random loss X has support on $[0, \infty)$ and $\mathbb{E}[X] < \infty$. To avoid tedious discussions and arguments, we simply suppose that the survival function $S_X(x)$ of X is continuous and decreasing on $[0, \infty)$ with $S_X(0) = 1$. Furthermore, we assume that the reinsurance premium is calculated by the expected value principle, namely, $P_I = (1 + \theta)\mathbb{E}[I(X)]$, where $\theta > 0$.

4.1 Reinsurance models taking into account the interests of both an insurer and a reinsurer

Assume the insurer and the reinsurer use the VaR with risk levels $0 < \alpha < 1$ and $0 < \beta < 1$, respectively, to measure their own losses. Without a reinsurance, the VaR of the insurer' loss is $\mathrm{VaR}_\alpha(X)$. With a reinsurance contract I, the VaR of the insurer's loss is $\mathrm{VaR}_\alpha(X - I(X) + P_I)$, and the insurer requires $\mathrm{VaR}_\alpha(X - I(X) + P_I) \leq \mathrm{VaR}_\alpha(X)$. Furthermore, the insurer wants the VaR to be reduced to a tolerated value L_1 so that

$$\mathrm{VaR}_\alpha(X - I(X) + P_I) \leq L_1, \tag{4.1}$$

where $L_1 > 0$ is the threshold representing the maximum VaR tolerated by the insurer after a reinsurance. Thus, it is reasonable to assume $L_1 \leq \mathrm{VaR}_\alpha(X)$.

On the other hand, the reinsurer also worries about his loss in selling the contract I and wants to set a threshold $L_2 > 0$ for the VaR of his loss so that

$$\mathrm{VaR}_\beta(I(X) - P_I) \leq L_2. \tag{4.2}$$

Note that $I(X) - X \leq 0 \leq P_I$. Thus, $I(X) - P_I \leq X$ and $\mathrm{VaR}_\beta(I(X) - P_I) \leq \mathrm{VaR}_\beta(X)$. Hence, it is reasonable to assume $L_2 \leq \mathrm{VaR}_\beta(X)$.

As the seller of the reinsurance contract I, the reinsurer expects to make a profit, namely, to have $I(X) \leq P_I$. Assume that the reinsurer wants to make a profit at least $L_3 \geq 0$ at a confidence level at least $0 < \gamma < 1$ in selling the reinsurance contract I, namely the profit target L_3 and the confidence level γ satisfy

$$\mathbb{P}\left(P_I - I(X) \geq L_3\right) = 1 - \mathbb{P}\left(I(X) > P_I - L_3\right) \geq \gamma. \tag{4.3}$$

To obtain feasible and applicable models for optimal reinsurance designs from the perspectives of both an insurer and a reinsurer, we have to make some assumptions on the relationships between the confidence level γ and each of the risk levels α and β, and the safety loading factor θ. In doing so, suppose $1 - \gamma \leq \beta$. Then, $\mathrm{VaR}_\beta\left(I(X) - P_I\right) \leq \mathrm{VaR}_{1-\gamma}\left(I(X) - P_I\right) \leq -L_3 \leq 0$, where the second inequality follows from (4.3). However, the risk level β is used to measure the maximum possible loss of the reinsurer. If $1-\gamma \leq \beta$, then the level β will lead to a non-positive VaR for his loss $I(X) - P_I$. Such a non-positive VaR cannot provide useful information for the reinsurer. Thus, we assume $\beta < 1-\gamma$. In addition, we assume $\alpha < 1-\gamma$ as well, since the risk levels α and β should be near in practice.

Furthermore, for a feasible contract $I \in \mathcal{I}$, note that $I(X)$ is a nonnegative random variable and $P_I = (1+\theta)\mathbb{E}[I(X)]$, thus by Markov's inequality, it is easy to see $\mathbb{P}\left(I(X) > P_I\right) \leq 1/(1+\theta)$ or equivalently $\mathbb{P}\left(I(X) \leq P_I\right) \geq \theta/(1+\theta)$, which implies that the reinsurer will make a profit, namely, $I(X) \leq P_I$, with a probability at least $\theta/(1+\theta)$. Thus, it is reasonable to assume $\gamma > \theta/(1+\theta)$ since L_3 is the profit target or the minimum profit desire for the reinsurer to sell a reinsurance contract and only a very high confidence level γ is acceptable for the reinsurer. Note that $\gamma > \theta/(1+\theta)$ is equivalent to $1-\gamma < 1/(1+\theta)$. Hence, the assumptions of $\alpha < 1-\gamma$ and $\beta < 1-\gamma$ imply $\alpha < 1/(1+\theta)$ and $\beta < 1/(1+\theta)$, respectively.

Throughout the chapter, we denote $a = \mathrm{VaR}_\alpha(X)$, $b = \mathrm{VaR}_\beta(X)$, $c = \mathrm{VaR}_{1-\gamma}(X)$, and $v_\theta = \mathrm{VaR}_{\frac{1}{1+\theta}}(X)$. Therefore, for any $I \in \mathcal{I}$, by the properties of the VaR, we have $\mathrm{VaR}_\alpha(X - I(X) + P_I) = a - I(a) + P_I$, $\mathrm{VaR}_\beta(I(X) - P_I) = I(b) - P_I$, and $\mathrm{VaR}_{1-\gamma}(I(X)) = I(c)$. It is easy to check that (4.1) is equivalent to $a - I(a) \leq L_1 - P_I$, (4.2) is equivalent to $I(b) \leq L_2 + P_I$, and (4.3) is equivalent to $I(c) \leq P_I - L_3$. Moreover, note that $\alpha \vee \beta < 1-\gamma < 1/(1+\theta)$ is equivalent to $v_\theta < c < a \wedge b$.

Thus, when the insurer and the reinsurer have the limits L_1 and L_2, respectively, on the VaRs of their own losses in a reinsurance contract, the set of the feasible reinsurance contracts acceptable by both the insurer and the reinsurer is

$$\mathcal{I}_1 = \left\{ I \in \mathcal{I} : \, I(b) - L_2 \leq P_I \leq I(a) - a + L_1 \right\}, \tag{4.4}$$

where $\mathcal{I}_1$ is obtained when the constraints (4.1) and (4.2) are imposed on $\mathcal{I}$.

Furthermore, when the insurer has the limit L_1 on the VaR of his loss and the reinsurer has the target L_3 on his profit in a reinsurance contract, the set of the feasible reinsurance contracts acceptable by both the insurer and the reinsurer is

$$\mathcal{I}_2 = \left\{ I \in \mathcal{I} : \, I(c) + L_3 \leq P_I \leq I(a) - a + L_1 \right\}, \tag{4.5}$$

where $\mathcal{I}_2$ is obtained when the constraints (4.1) and (4.3) are imposed on $\mathcal{I}$.

The desired sets $\mathcal{I}_1$ and $\mathcal{I}_2$ may be empty. We have to impose some restrictions on L_1, L_2, and L_3 so that $\mathcal{I}_1$ and $\mathcal{I}_2$ are non-empty. First, for any $I \in \mathcal{I}_1$, we have $L_1 + L_2 \geq a - I(a) + P_I + I(b) - P_I = a + I(b) - I(a)$. Furthermore, by the 1-Lipschitz continuity of

I, we have $I(b) - I(a) \geq 0$ if $b > a$ and $I(b) - I(a) \geq b - a$ if $a > b$. Hence, $L_1 + L_2 \geq a \wedge b$. Moreover, we assume

$$v_\theta + (1+\theta) \int_{v_\theta}^{a} S_X(x)\mathrm{d}x \leq L_1. \tag{4.6}$$

This condition will guarantee that $\mathcal{I}_1$ is non-empty as showed in Lemma 4.2.2.

Next, for any $I \in \mathcal{I}_2$, because $a \geq c$ and I is 1-Lipschitz continuous, we have $a + P_I - L_1 - P_I + L_3 \leq I(a) - I(c) \leq a - c$, and thus $c \leq L_1 - L_3$.

Furthermore, we assume

$$(1+\theta)\left(\int_0^{v_\theta} + \int_c^{\infty}\right) S_X(x)\mathrm{d}x - v_\theta \geq L_3. \tag{4.7}$$

The conditions (4.6) and (4.7) will guarantee $\mathcal{I}_2$ to be non-empty as proved in Lemma 4.3.5.

When $\mathcal{I}_i$, $i = 1, 2$, is the set of feasible reinsurance contracts acceptable by both the insurer and the reinsurer, from the insurer's perspective, an optimal reinsurance contract is a solution to the optimization problem of

$$\min_{I \in \mathcal{I}_i} \mathrm{VaR}_\alpha\left(X - I(X) + P_I\right), \tag{4.8}$$

while from the reinsurer's perspective, an optimal reinsurance contract is a solution to the optimization problem of

$$\min_{I \in \mathcal{I}_i} \mathrm{VaR}_\beta\left(I(X) - P_I\right). \tag{4.9}$$

Instead of solving Problems (4.8) and (4.9) separately, we consider the unified minimization problem of

$$\min_{I \in \mathcal{I}_i} V(I), \tag{4.10}$$

where the objective function

$$\begin{aligned} V(I) &= \lambda \mathrm{VaR}_\alpha\left(X - I(X) + P_I\right) + (1-\lambda)\mathrm{VaR}_\beta\left(I(X) - P_I\right) \\ &= \lambda a + (2\lambda - 1)P_I - \lambda I(a) + (1-\lambda)I(b), \end{aligned}$$

is the convex combination of the VaRs of the insurer's loss and the reinsurer's loss, with $\lambda \in [0,1]$ a weighting factor. When $\lambda = 0$, $V(I) = \mathrm{VaR}_\beta\left(I(X) - P_I\right)$ and Problem (4.10) is reduced to Problem (4.9). When $\lambda = 1$, $V(I) = \mathrm{VaR}_\alpha\left(X - I(X) + P_I\right)$ and Problem (4.10) is reduced to Problem (4.8). Thus, Problems (4.8) and (4.9) can be viewed as special cases of Problem (4.10).

When $a = b$, the objective function $V(I)$ becomes

$$V(I) = \lambda a + (1 - 2\lambda)(I(a) - P_I) = (1 - \lambda)a + (2\lambda - 1)(a - I(a) + P_I),$$

which implies that Problem (4.10) is reduced to either Problem (4.8) when $1/2 < \lambda \leq 1$ or Problem (4.9) when $0 \leq \lambda < 1/2$. However, these two problems are covered in Problem (4.10) by setting $\lambda = 1$ and $\lambda = 0$, respectively. Thus, we assume $a \neq b$.

Furthermore, when $\lambda = 1/2$, the objective function $V(I)$ becomes

$$V(I) = \frac{a}{2} + \frac{1}{2}\left(I(b) - I(a)\right).$$

Thus, Problem (4.10) is reduced to $\min_{I \in \mathcal{I}_i} \{I(b) - I(a)\}$, $i = 1, 2$. Note that the 1-Lipschitz property of I implies that $0 \leq I(b) - I(a) \leq b - a$ for $a < b$ and $I(b) - I(a) \geq -(a - b)$ for $a > b$. Hence, $\min_{I \in \mathcal{I}_i} \{I(b) - I(a)\} = -(a - b)^+$. Thus, the optimal contract I^* to the problem of $\min_{I \in \mathcal{I}_i} \{I(b) - I(a)\}$ and hence to Problem (4.10) is any contract $I^* \in \mathcal{I}_i$ satisfying $I^*(a) - I^*(b) = (a - b)^+$. We will see in Remarks 4.2.1 and 4.3.1 that such optimal contracts I^* exist in $\mathcal{I}_i$ for $i = 1, 2$, and thus Problem (4.10) is solved for $\lambda = 1/2$. Hence, we assume $\lambda \neq 1/2$.

In summary, in the rest of this chapter, we assume that the following conditions hold:

$$\begin{cases} \lambda \neq \frac{1}{2},\ a \neq b,\ L_3 + c \leq L_1 \leq a,\ L_2 \leq b,\ 0 < v_\theta < c < a \wedge b \leq L_1 + L_2, \\ \text{and the inequalities (4.6) and (4.7) hold.} \end{cases} \tag{4.11}$$

Next, we will solve Problem (4.10) for $i = 1, 2$ in Sections 4.2 and 4.3, respectively.

4.2 Constraints on both an insurer's loss and a reinsurer's loss

In this section, we will solve Problem (4.10) for $i = 1$, namely, to solve the minimization problem of

$$\min_{I \in \mathcal{I}_1} V(I). \tag{4.12}$$

In this problem, $V(I) = \lambda a + (2\lambda - 1)P_I - \lambda I(a) + (1-\lambda)I(b)$ and $\mathcal{I}_1$ is the set of feasible reinsurance contracts acceptable by both the insurer and the reinsurer. The definition of $\mathcal{I}_1$ also describes the constraints on the VaRs of both an insurer's loss and a reinsurer's loss. A reinsurance contract I is said to be acceptable if $I \in \mathcal{I}_1$.

First, we introduce some notation. Define the two types of feasible contract $I^m_{\xi_a,\xi_b}$ and $I^M_{\xi_a,\xi_b}$ in $\mathcal{I}$ for some pairs of (ξ_a, ξ_b) as follows:

(1) If $a < b$, for each pair $(\xi_a, \xi_b) \in [0, a] \times [0, b]$ and $\xi_a \leq \xi_b$, define

$$\begin{aligned} I^m_{\xi_a,\xi_b}(x) &= (x - a + \xi_a)^+ - (x-a)^+ + (x - (b - \xi_b + \xi_a))^+ - (x-b)^+, \\ I^M_{\xi_a,\xi_b}(x) &= x - (x - \xi_a)^+ + (x-a)^+ - (x - (a + \xi_b - \xi_a))^+ + (x-b)^+. \end{aligned}$$

(2) If $a > b$, for each pair $(\xi_a, \xi_b) \in [0, a] \times [0, b]$ and $\xi_a \geq \xi_b$, define

$$\begin{aligned} I^m_{\xi_a,\xi_b}(x) &= (x - b + \xi_b)^+ - (x-b)^+ + (x - (a - \xi_a + \xi_b))^+ - (x-b)^+, \\ I^M_{\xi_a,\xi_b}(x) &= x - (x - \xi_b)^+ + (x-b)^+ - (x - (b + \xi_a - \xi_b))^+ + (x-a)^+. \end{aligned}$$

Since $I^m_{\xi_a,\xi_b}(0) = 0$ and $\lim_{x\to\infty} S_X(x) = 0$, we have

$$\begin{aligned} P_{I^m_{\xi_a,\xi_b}} &= (1+\theta)\mathbb{E}\left[I^m_{\xi_a,\xi_b}(X)\right] = (1+\theta)\int_0^\infty I^m_{\xi_a,\xi_b}(x)\mathrm{d}F_X(x) \\ &= -(1+\theta)\int_0^\infty I^m_{\xi_a,\xi_b}(x)\mathrm{d}S_X(x) = (1+\theta)\int_0^\infty S_X(x)\mathrm{d}I^m_{\xi_a,\xi_b}(x) \\ &= (1+\theta)\left(\int_{a\wedge b - \xi_a\wedge\xi_b}^{a\wedge b} + \int_{a\vee b - |\xi_b - \xi_a|}^{a\vee b}\right) S_X(x)\mathrm{d}x. \end{aligned}$$

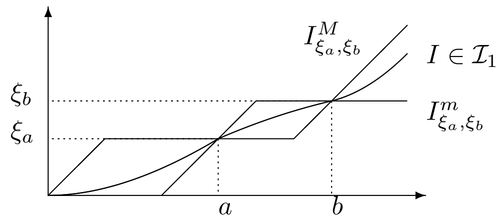
Similarly, we have

$$P_{I^M_{\xi_a,\xi_b}} = (1+\theta)\mathbb{E}\left[I^M_{\xi_a,\xi_b}(X)\right] = (1+\theta)\left(\int_0^{\xi_a\wedge\xi_b} + \int_{a\wedge b}^{a\wedge b + |\xi_b - \xi_a|} + \int_{a\vee b}^\infty\right) S_X(x)\mathrm{d}x.$$

It is easy to verify that for any $I \in \mathcal{I}_1$ satisfying $I(a) = \xi_a$ and $I(b) = \xi_b$, we have $I^m_{\xi_a,\xi_b}(x) \leq I(x) \leq I^M_{\xi_a,\xi_b}(x)$ for all $x \geq 0$ as illustrated by Figure 4.1 and thus $P_{I^m_{\xi_a,\xi_b}} \leq P_I \leq P_{I^M_{\xi_a,\xi_b}}$.

Next, we define the set $\Xi_{a,b} \subset [0, a] \times [0, b]$ as follows:

Figure 4.1: Relation between arbitrary $I \in \mathcal{I}_1$ and the pair $(I^m_{\xi_a,\xi_b}, I^M_{\xi_a,\xi_b})$.



(a) When $a < b$, $\Xi_{a,b}$ is the set of all pairs (ξ_a, ξ_b) satisfying

$$\xi_a \le \xi_b \le \xi_a + b \wedge (L_1 + L_2) - a, \tag{4.13}$$

$$\xi_b - L_2 \le P_{I^M_{\xi_a,\xi_b}} = (1+\theta)\left(\int_0^{\xi_a} + \int_a^{a+\xi_b-\xi_a} + \int_b^{\infty}\right) S_X(x)\mathrm{d}x, \tag{4.14}$$

$$L_1 - a + \xi_a \ge P_{I^m_{\xi_a,\xi_b}} = (1+\theta)\left(\int_{a-\xi_a}^{a} + \int_{b-\xi_b+\xi_a}^{b}\right) S_X(x)\mathrm{d}x. \tag{4.15}$$

(b) When $a > b$, $\Xi_{a,b}$ is the set of all pairs (ξ_a, ξ_b) satisfying

$$\xi_b + (a - L_1 - L_2)^+ \le \xi_a \le \xi_b + a - b, \tag{4.16}$$

$$\xi_b - L_2 \le P_{I^M_{\xi_a,\xi_b}} = (1+\theta)\left(\int_0^{\xi_b} + \int_b^{b+\xi_a-\xi_b} + \int_a^{\infty}\right) S_X(x)\mathrm{d}x, \tag{4.17}$$

$$L_1 - a + \xi_a \ge P_{I^m_{\xi_a,\xi_b}} = (1+\theta)\left(\int_{b-\xi_b}^{b} + \int_{a-\xi_a+\xi_b}^{a}\right) S_X(x)\mathrm{d}x. \tag{4.18}$$

To solve Problem (4.12), we introduce the auxiliary functions g_1, g_2, and g_3 and discuss their properties in the following proposition.

Proposition 4.2.1 *(a) Define $g_1(\xi) = \xi - (1+\theta)\int_{a-\xi}^{a} S_X(x)\mathrm{d}x$ for $\xi \in [0, a]$. Then, g_1 is continuous, increasing on $[0,\, a-v_\theta)$, strictly decreasing on $(a-v_\theta,\, a]$, and $\max_{\xi\in[0,\,a]} g_1(\xi) = g_1(a - v_\theta)$.*

(b) Define $g_2(\xi) = \xi - (1+\theta)\left(\int_0^{\xi} + \int_b^{\infty}\right) S_X(x)\mathrm{d}x$ for $\xi \in [0,\, a \wedge b]$. Then, g_2 is continuous, strictly decreasing on $[0, v_\theta)$, increasing on $(v_\theta, a \wedge b]$, and $\min_{\xi\in[0,\,a\wedge b]} g_2(\xi) = g_2(v_\theta)$.

(c) Define $g_3(\xi) = \xi - (1+\theta)\int_{b-\xi}^{a} S_X(x)\mathrm{d}x$ *for* $\xi \in [0, b]$*. Then,* g_3 *is continuous, increasing on* $[0, b-v_\theta)$*, strictly decreasing on* $(b-v_\theta, b]$*, and* $\max_{\xi\in[0,b]} g_3(\xi) = g_3(b-v_\theta)$*.*

(d) Assume $a < b$*. Then* $g_2(\xi_a) < g_1(\xi_a)$ *for any* $\xi_a \in [0, a]$*. In addition,* $P_{I^M_{\xi_a,\xi_b}}$*,* $P_{I^m_{\xi_a,\xi_b}}$*, and* $\xi_b - P_{I^M_{\xi_a,\xi_b}}$ *are continuous and strictly increasing in* $\xi_b \in [\xi_a, \xi_a + b - a]$*.*

(e) Assume $a > b$*. Then* $g_2(\xi_b) < g_3(\xi_b)$ *for any* $\xi_b \in [0, b]$*. In addition,* $P_{I^M_{\xi_a,\xi_b}}$*,* $P_{I^m_{\xi_a,\xi_b}}$*, and* $\xi_a - P_{I^m_{\xi_a,\xi_b}}$ *are continuous and strictly increasing in* $\xi_a \in [\xi_b, \xi_b + a - b]$*.*

Lemma 4.2.2 *The following three statements are equivalent:*

(i) Inequality (4.6) holds.

(ii) $\mathcal{I}_1 \neq \emptyset$*.*

(iii) $\Xi_{a,b} \neq \emptyset$*.*

In addition, (4.6) implies

$$v_\theta - (1+\theta)\left(\int_0^{v_\theta} + \int_b^{\infty}\right) S_X(x)\mathrm{d}x \le L_2. \tag{4.19}$$

Lemma 4.2.3 *Problem (4.12) has the same minimal value as the minimization problem*

$$\min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a, \xi_b) \tag{4.20}$$

in the sense that $\min_{I\in\mathcal{I}_1} V(I) = \min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a,\xi_b)$*, where,* $v(\xi_a,\xi_b) = \lambda a + (2\lambda - 1)P_{\xi_a,\xi_b} - \lambda\xi_a + (1-\lambda)\xi_b$ *and*

$$P_{\xi_a,\xi_b} = \begin{cases} (L_1 - a + \xi_a) \wedge P_{I^M_{\xi_a,\xi_b}}, & \text{if } 0 \le \lambda < \frac{1}{2}, \\ (\xi_b - L_2) \vee P_{I^m_{\xi_a,\xi_b}}, & \text{if } \frac{1}{2} < \lambda \le 1. \end{cases} \tag{4.21}$$

Moreover, let $(\xi_a^*, \xi_b^*) \in \Xi_{a,b}$ *be the minimizer of Problem (4.20). Then, a contract* I^* *of the form*

$$\begin{aligned} I^*(x) = &(x-d_1)^+ - (x-(d_1+\xi_a^*\wedge\xi_b^*))^+ + (x-d_2)^+ - (x-(d_2+|\xi_b^*-\xi_a^*|))^+ \\ &+ (x-d_3)^+ \end{aligned} \tag{4.22}$$

for some $(d_1, d_2, d_3) \in [0,\, a\wedge b - \xi_a^*\wedge\xi_b^*] \times [a\wedge b,\, a\vee b - |\xi_b^* - \xi_a^*|] \times [a\vee b,\, \infty]$*, satisfying* $P_{I^*} = P_{\xi_a^*,\xi_b^*}$*, is an optimal solution to Problem (4.12).*

Lemma 4.2.3 reduces the infinite-dimensional optimization problem (4.12) to a two-dimensional optimization problem (4.20). In the following two theorems, we give the explicit expressions of (ξ_a^*, ξ_b^*) and (d_1, d_2, d_3) for the optimal solution I^* presented in (4.22).

Theorem 4.2.4 *Suppose $a < b$, then Problem (4.20) has minimizer (ξ_a^*, ξ_b^*) with $\xi_a^* = \xi_b^*$ and the optimal solution to Problem (4.12), denoted by I^*, is given as follows.*

(a) In the case $0 \le \lambda < 1/2$:

(i) If $g_1(v_\theta) \ge a - L_1$, then $\xi_a^* = v_\theta$ and

$$I^*(x) = (x - d_1)^+ - (x - d_1 - v_\theta)^+ + (x - d_3)^+,$$

for some $d_1 \in [0, a - v_\theta]$ and $d_3 \in [b, \infty]$ such that $P_{I^*} = v_\theta - (a - L_1) \vee g_2(v_\theta)$.

(ii) If $g_1(v_\theta) < a - L_1$, then there exists $\xi_1 \in [v_\theta \wedge (a - v_\theta), v_\theta \vee (a - v_\theta)]$ such that $g_1(\xi_1) = a - L_1$. Moreover, $\xi_a^* = \xi_1$ and

$$I^*(x) = (x - a + \xi_1)^+ - (x - a)^+.$$

(b) In the case $1/2 < \lambda \le 1$:

(i) If $g_2(a - v_\theta) \le L_2$, then $\xi_a^* = a - v_\theta$ and

$$I^*(x) = (x - d_1)^+ - (x - d_1 - a + v_\theta)^+ + (x - d_3)^+,$$

for some $d_1 \in [0, v_\theta]$ and $d_3 \in [b, \infty]$ such that $P_{I^*} = a - v_\theta - L_2 \wedge g_1(a - v_\theta)$.

(ii) If $g_2(a - v_\theta) > L_2$, then there exists $\xi_2 \in [v_\theta \wedge (a - v_\theta), v_\theta \vee (a - v_\theta)]$ such that $g_2(\xi_2) = L_2$. Moreover, $\xi_a^* = \xi_2$ and

$$I^*(x) = x - (x - \xi_2)^+ + (x - b)^+.$$

Theorem 4.2.5 *Suppose $a > b$, then Problem (4.20) has minimizer (ξ_a^*, ξ_b^*) with $\xi_a^* = \xi_b^* + a - b$ and the optimal solution to Problem (4.12), denoted by I^*, is given as follows.*

(a) In the case $0 \le \lambda < 1/2$:

(i) If $g_3(v_\theta) \ge b - L_1$, then $\xi_b^* = v_\theta$ and

$$I^*(x) = (x - d_1)^+ - (x - d_1 - v_\theta)^+ + (x - b)^+ - (x - d_3)^+,$$

for some $d_1 \in [0, b - v_\theta]$ and $d_3 \in [a, \infty]$ such that $P_{I^*} = v_\theta - (b - L_1) \vee g_2(v_\theta)$.

(ii) *If* $g_3(v_\theta) < b - L_1$*, then there exists* $\xi_3 \in [v_\theta \wedge (b - v_\theta), v_\theta \vee (b - v_\theta)]$ *such that* $g_3(\xi_3) = b - L_1$*. Moreover,* $\xi_b^* = \xi_3$ *and*

$$I^*(x) = (x - b + \xi_3)^+ - (x - a)^+.$$

(b) *In the case of* $1/2 < \lambda \leq 1$*:*

(i) *If* $g_2(b - v_\theta) \leq L_2$*, then* $\xi_b^* = b - v_\theta$ *and*

$$I^*(x) = (x - d_1)^+ - (x - d_1 - b + v_\theta)^+ + (x - b)^+ - (x - d_3)^+,$$

for some $d_1 \in [0, v_\theta]$ *and* $d_3 \in [a, \infty]$ *such that* $P_{I^*} = b - v_\theta - L_2 \wedge g_3(b - v_\theta)$*.*

(ii) *If* $g_2(b - v_\theta) > L_2$*, then there exists* $\xi_4 \in [v_\theta \wedge (b - v_\theta), v_\theta \vee (b - v_\theta)]$ *such that* $g_2(\xi_4) = L_2$*. Moreover,* $\xi_b^* = \xi_4$ *and*

$$I^*(x) = x - (x - \xi_4)^+ + (x - b)^+.$$

Remark 4.2.1 *From Theorems 4.2.5 and 4.2.4, it is easy to see that the optimal solution* I^* *can be separated into two different cases of* $0 \leq \lambda < 1/2$ *and* $1/2 < \lambda \leq 1$*, but the parameters in each case don't depend on* λ*.*

By the proofs of Theorems 4.2.4 and 4.2.5, we know that the optimal contracts I^* *in Theorems 4.2.4 and 4.2.5 satisfy* $I^*(a) - I^*(b) = (a - b)^+$*, and hence the optimal solutions* I^* *in Theorems 4.2.4 and 4.2.5 are also the solutions to Problem (4.12) when* $\lambda = 1/2$*.* □

4.3 Constraints on an insurer's loss and a reinsurer's profit

In this section, we solve Problem (4.10) for $i = 2$, namely, we solve the minimization problem

$$\min_{I \in \mathcal{I}_2} V(I). \tag{4.23}$$

In this problem, $V(I) = \lambda a + (2\lambda - 1)P_I - \lambda I(a) + (1 - \lambda)I(b)$ and $\mathcal{I}_2$ is the set of feasible reinsurance contracts acceptable by both the insurer and the reinsurer. The definition of $\mathcal{I}_2$ also describes the constraints on the VaR of the insurer's loss and on the reinsurer's profit. A reinsurance contract I is said to be acceptable if $I \in \mathcal{I}_2$.

It is easy to check that for any given $(\xi_c, \xi_a, \xi_b) \in [0, c] \times [0, a] \times [0, b]$, if $I \in \mathcal{I}$ satisifes $I(c) = \xi_c$, $I(a) = \xi_a$, and $I(b) = \xi_b$, then $I^m_{\xi_c,\xi_a,\xi_b}(x) \le I(x) \le I^M_{\xi_c,\xi_a,\xi_b}(x)$ for all $x \ge 0$ and $P_{I^m_{\xi_c,\xi_a,\xi_b}} \le P_I \le P_{I^M_{\xi_c,\xi_a,\xi_b}}$, where

$$\begin{aligned} I^m_{\xi_c,\xi_a,\xi_b}(x) =& (x - c + \xi_c)^+ - (x - c)^+ + (x - (a \wedge b - \xi_a \wedge \xi_b + \xi_c))^+ - (x - a \wedge b)^+ \\ &+ (x - (a \vee b - |\xi_a - \xi_b|))^+ - (x - a \vee b)^+, \\ I^M_{\xi_c,\xi_a,\xi_b}(x) =& x - (x - \xi_c)^+ + (x - c)^+ - (x - (c + \xi_a \wedge \xi_b - \xi_c))^+ \\ &+ (x - a \wedge b)^+ - (x - (a \wedge b + |\xi_a - \xi_b|))^+ + (x - a \vee b)^+, \end{aligned}$$

are two feasible reinsurance contracts in $\mathcal{I}$.

To solve Problem (4.23), we introduce auxiliary functions h_i for $i = 1, ..., 7$, $A^M_{\xi_c}$, A_{ξ_c}, $A^m_{\xi_c}$, $B^M_{\xi_c}$, and $B^m_{\xi_c}$, and discuss their properties in the following three propositions.

Proposition 4.3.1 *Assume $a \neq b$.*

(a) Define $h_1(\xi_c) = (1 + \theta) \left(\int_0^{\xi_c} + \int_c^{\infty}\right) S_X(x)\mathrm{d}x - \xi_c$ for $\xi_c \in [0, c]$. Then $h_1(\xi_c)$ is continuous, concave, strictly increasing on $[0, v_\theta)$, decreasing on $(v_\theta, c]$, and $\max_{\xi_c \in [0,\, c]} h_1(\xi_c) = h_1(v_\theta)$.

(b) Define $h_2(\xi_c) = (1 + \theta) \int_{c-\xi_c}^{a} S_X(x)\mathrm{d}x - \xi_c$ for $\xi_c \in [0, c]$. Then $h_2(\xi_c)$ is continuous, convex, decreasing on $[0, c - v_\theta)$, strictly increasing on $(c - v_\theta, c]$, and $\min_{\xi_c \in [0,\, c]} h_2(\xi_c) = h_2(c - v_\theta)$. Moreover, $h_2(\xi_c) < h_1(\xi_c)$ for $\xi_c \in [0, c]$.

Proposition 4.3.2 *Assume $a < b$.*

(a) Functions $P_{I^M_{\xi_c,\xi_a,\xi_b}}$, $P_{I^m_{\xi_c,\xi_a,\xi_b}}$, $\xi_b - P_{I^M_{\xi_c,\xi_a,\xi_b}}$, and $\xi_b - P_{I^m_{\xi_c,\xi_a,\xi_b}}$, are continuous and strictly increasing in $\xi_b \in [\xi_a, \xi_a + b - a]$.

(b) Given $\xi_c \in [0, c]$, define $A^M_{\xi_c}(\xi_a) = P_{I^M_{\xi_c,\xi_a,\xi_a+b-a}}$ and $A^m_{\xi_c}(\xi_a) = P_{I^m_{\xi_c,\xi_a,\xi_a}}$, for $\xi_a \in [\xi_c, \xi_c + a - c]$, and $A_{\xi_c}(\xi_a) = P_{I^M_{\xi_c,\xi_a,\xi_a}}$, for $\xi_a \in [\xi_c, \xi_c + b - c]$. Then all the functions $A^M_{\xi_c}(\xi_a)$, $A^m_{\xi_c}(\xi_a)$, $\xi_a - A^M_{\xi_c}(\xi_a)$ and $\xi_a - A^m_{\xi_c}(\xi_a)$ are continuous and strictly increasing in $\xi_a \in [\xi_c, \xi_c + a - c]$, and $A_{\xi_c}(\xi_a)$ and $\xi_a - A_{\xi_c}(\xi_a)$ are continuous and strictly increasing in $\xi_a \in [\xi_c, \xi_c + b - c]$.

(c) Define $h_3(\xi_c) = A_{\xi_c}(\xi_c + a - c) - \xi_c$ for $\xi_c \in [0, c]$. Then $h_3(\xi_c)$ is continuous, concave, strictly increasing on $[0, v_\theta)$, decreasing on $(v_\theta, c]$, and $\max_{\xi_c \in [0,\, c]} h_3(\xi_c) = h_3(v_\theta)$.

(d) Define $h_4(\xi_c) = A^m_{\xi_c}(\xi_c + a - L_1 + L_3) - \xi_c$ for $\xi_c \in [0, c]$. Then $h_4(\xi_c)$ is continuous, convex, decreasing on $[0, c - v_\theta)$, strictly increasing on $(c - v_\theta, c]$, and $\min_{\xi_c \in [0,\, c]} h_4(\xi_c) = h_4(c - v_\theta)$.

(e) Define $h_5(\xi_c) = A_{\xi_c}(\xi_c + a - L_1 + L_3) - \xi_c$ for $\xi_c \in [0, c]$. Then $h_5(\xi_c)$ is continuous, concave, strictly increasing on $[0, v_\theta)$, decreasing on $(v_\theta, c]$, and $\max_{\xi_c \in [0,c]} h_5(\xi_c) = h_5(v_\theta)$.

(f) Given $\xi_c \in [0, c]$, it holds that $A^m_{\xi_c}(\xi_a) < A_{\xi_c}(\xi_a) < A^M_{\xi_c}(\xi_a)$ for $\xi_a \in [\xi_c, \xi_c + a - c]$. Furthermore, it holds that $h_4(\xi_c) < h_5(\xi_c) \le h_3(\xi_c)$ for $\xi_c \in [0, c]$. In addition, $h_5(\xi_c) = h_3(\xi_c)$ if and only if $c = L_1 - L_3$.

Proposition 4.3.3 *Assume $a > b$.*

(a) Functions $P_{I^M_{\xi_c,\xi_a,\xi_b}}$, $P_{I^m_{\xi_c,\xi_a,\xi_b}}$, $\xi_a - P_{I^M_{\xi_c,\xi_a,\xi_b}}$, and $\xi_a - P_{I^m_{\xi_c,\xi_a,\xi_b}}$, are continuous and strictly increasing in $\xi_a \in [\xi_b, \xi_b + a - b]$.

(b) Given $\xi_c \in [0, c]$, define $B^M_{\xi_c}(\xi_b) = P_{I^M_{\xi_c,\xi_b+a-b,\xi_b}}$ and $B^m_{\xi_c}(\xi_b) = P_{I^m_{\xi_c,\xi_b+a-b,\xi_b}}$ for $\xi_b \in [\xi_c, \xi_c + b - c]$. Then all the functions $B^M_{\xi_c}(\xi_b)$, $B^m_{\xi_c}(\xi_b)$, $\xi_b - B^M_{\xi_c}(\xi_b)$, and $\xi_b - B^m_{\xi_c}(\xi_b)$, are continuous and strictly increasing in $\xi_b \in [\xi_c, \xi_c + b - c]$.

(c) Define $h_6(\xi_c) = B^m_{\xi_c}(\xi_c + (b - L_1 + L_3)^+) - \xi_c$ for $\xi_c \in [0, c]$. Then $h_6(\xi_c)$ is continuous, convex, decreasing on $[0, c - v_\theta)$, strictly increasing on $(c - v_\theta, c]$, and $\min_{\xi_c \in [0,c]} h_6(\xi_c) = h_6(c - v_\theta)$.

(d) Define $h_7(\xi_c) = B^M_{\xi_c}(\xi_c + (b - L_1 + L_3)^+) - \xi_c$ for $\xi_c \in [0, c]$. Then $h_7(\xi_c)$ is continuous, concave, strictly increasing on $[0, v_\theta)$, decreasing on $(v_\theta, c]$, and $\max_{\xi_c \in [0,c]} h_7(\xi_c) = h_7(v_\theta)$.

(e) Given $\xi_c \in [0, c]$, it holds that $B^m_{\xi_c}(\xi_b) < B^M_{\xi_c}(\xi_b)$ for $\xi_b \in [\xi_c, \xi_c + b - c]$. Furthermore, it holds that $h_6(\xi_c) < h_7(\xi_c)$ for $\xi_c \in [0, c]$.

Furthermore, we need to define the following sets. Let $\Xi_{c,a,b}$ be the set of all $(\xi_c, \xi_a, \xi_b) \in [0, c] \times [0, a] \times [0, b]$ such that

$$\xi_c + (a \wedge b + L_3 - L_1)^+ \le \xi_a \wedge \xi_b \le \xi_a \vee \xi_b, \tag{4.24}$$

$$\xi_c + L_3 \le P_{I^M_{\xi_c,\xi_a,\xi_b}}, \tag{4.25}$$

$$L_1 - a + \xi_a \ge P_{I^m_{\xi_c,\xi_a,\xi_b}}. \tag{4.26}$$

Let Ξ_c be the set of all $\xi_c \in [0, c]$ such that

$$L_3 + \xi_c \le (1 + \theta) \left(\int_0^{\xi_c} + \int_c^{\infty} \right) S_X(x) \mathrm{d}x, \tag{4.27}$$

$$L_1 - c + \xi_c \ge (1 + \theta) \int_{c-\xi_c}^{a} S_X(x) \mathrm{d}x. \tag{4.28}$$

For each $\xi_c \in \Xi_c$, if $a < b$, then let Ξ_{a,ξ_c} be the set of all $\xi_a \in [\xi_c + a + L_3 - L_1,\ \xi_c + a - c]$ such that

$$\xi_c + L_3 \le (1+\theta)\left(\int_0^{\xi_c} + \int_c^{c+\xi_a-\xi_c} + \int_a^{\infty}\right) S_X(x)\mathrm{d}x, \tag{4.29}$$

$$a - L_1 \le \xi_a - (1+\theta)\left(\int_{c-\xi_c}^{c} + \int_{a-\xi_a+\xi_c}^{a}\right) S_X(x)\mathrm{d}x, \tag{4.30}$$

and if $b < a$, let Ξ_{b,ξ_c} be the set of all $\xi_b \in [\xi_c + (b + L_3 - L_1)^+,\ \xi_c + b - c]$ such that

$$\xi_c + L_3 \le (1+\theta)\left(\int_0^{\xi_c} + \int_c^{c+\xi_b-\xi_c} + \int_b^{\infty}\right) S_X(x)\mathrm{d}x, \tag{4.31}$$

$$b - L_1 \le \xi_b - (1+\theta)\left(\int_{c-\xi_c}^{c} + \int_{b-\xi_b+\xi_c}^{a}\right) S_X(x)\mathrm{d}x. \tag{4.32}$$

If $a < b$, for each $(\xi_c, \xi_a) \in \Xi_c \times \Xi_{a,\xi_c}$, let Ξ_{b,ξ_c,ξ_a} be the set of all $\xi_b \in [\xi_a, \xi_a + b - a]$ such that $(\xi_c, \xi_a, \xi_b) \in \Xi_{c,a,b}$. If $a > b$, for each $(\xi_c, \xi_b) \in \Xi_c \times \Xi_{b,\xi_c}$, let Ξ_{a,ξ_c,ξ_b} be the set of all $\xi_a \in [\xi_b, \xi_b + a - b]$ such that $(\xi_c, \xi_a, \xi_b) \in \Xi_{c,a,b}$.

Proposition 4.3.4 *All the sets Ξ_c, Ξ_{a,ξ_c}, Ξ_{b,ξ_c}, Ξ_{b,ξ_c,ξ_a}, and Ξ_{a,ξ_c,ξ_b}, are closed intervals and can be expressed as follows.*

(a) The set $\Xi_c = [\xi_c^m,\ \xi_c^M]$ for some $0 \le \xi_c^m \le \xi_c^M \le c$.

(b) When $a < b$, given $\xi_c \in \Xi_c$, the set $\Xi_{a,\xi_c} = [\xi_a^m(\xi_c),\ \xi_a^M(\xi_c)]$ for some $\xi_c + a + L_3 - L_1 \le \xi_a^m(\xi_c) \le \xi_a^M(\xi_c) \le \xi_c + a - c$, and given $(\xi_c, \xi_a) \in \Xi_c \times \Xi_{a,\xi_c}$, the set $\Xi_{b,\xi_c,\xi_a} = [\xi_b^m(\xi_c, \xi_a),\ \xi_b^M(\xi_c, \xi_a)]$ for some $\xi_a \le \xi_b^m(\xi_c, \xi_a) \le \xi_b^M(\xi_c, \xi_a) \le \xi_a + b - a$.

(c) When $a > b$, given $\xi_c \in \Xi_c$, the set $\Xi_{b,\xi_c} = [\xi_b^m(\xi_c),\ \xi_b^M(\xi_c)]$ for some $\xi_c + (b + L_3 - L_1)^+ \le \xi_b^m(\xi_c) \le \xi_b^M(\xi_c) \le \xi_c + b - c$, and given $(\xi_c, \xi_b) \in \Xi_c \times \Xi_{b,\xi_c}$, the set $\Xi_{a,\xi_c,\xi_b} = [\xi_a^m(\xi_c, \xi_b),\ \xi_a^M(\xi_c, \xi_b)]$ for some $\xi_b \le \xi_a^m(\xi_c, \xi_b) \le \xi_a^M(\xi_c, \xi_b) \le \xi_b + a - b$.

Lemma 4.3.5 *The following three statements are equivalent:*

(i) Inequalities (4.6) and (4.7) hold.

(ii) $\mathcal{I}_2 \neq \emptyset$.

(iii) $\Xi_{c,a,b} \neq \emptyset$.

Lemma 4.3.6 *Problem (4.23) has the same minimal value as the minimization problem*

$$\min_{(\xi_c,\xi_a,\xi_b)\in\Xi_{c,a,b}} w(\xi_c,\xi_a,\xi_b) \tag{4.33}$$

in the sense that $\min_{I\in\mathcal{I}_2} V(I) = \min_{(\xi_c,\xi_a,\xi_b)\in\Xi_{c,a,b}} w(\xi_c,\xi_a,\xi_b)$, *where* $w(\xi_c,\xi_a,\xi_b) = \lambda a + (2\lambda-1)P_{\xi_c,\xi_a,\xi_b} - \lambda\xi_a + (1-\lambda)\xi_b$ *and*

$$P_{\xi_c,\xi_a,\xi_b} = \begin{cases} (L_1 - a + \xi_a) \wedge P_{I^M_{\xi_c,\xi_a,\xi_b}}, & \text{if} \quad 0 \le \lambda < \frac{1}{2}, \\ (\xi_c + L_3) \vee P_{I^m_{\xi_c,\xi_a,\xi_b}}, & \text{if} \quad \frac{1}{2} < \lambda \le 1. \end{cases} \tag{4.34}$$

To solve the three-dimensional problem (4.33), we consider the following three-step minimization problem:

$$\begin{cases} \min_{\xi_c\in\Xi_c}\left\{\min_{\xi_a\in\Xi_{a,\xi_c}}\left[\min_{\xi_b\in\Xi_{b,\xi_c,\xi_a}} w(\xi_c,\xi_a,\xi_b)\right]\right\}, & \text{if } a<b, \\ \min_{\xi_c\in\Xi_c}\left\{\min_{\xi_b\in\Xi_{b,\xi_c}}\left[\min_{\xi_a\in\Xi_{a,\xi_c,\xi_b}} w(\xi_c,\xi_a,\xi_b)\right]\right\}, & \text{if } a>b. \end{cases} \tag{4.35}$$

In doing so, we define the minimizers of Problem (4.35) and the corresponding functions as follows.

For $a<b$, define $\min_{\xi_b\in\Xi_{b,\xi_c,\xi_a}} w(\xi_c,\xi_a,\xi_b) = w(\xi_c,\xi_a,\xi_b^*(\xi_c,\xi_a)) = w_2(\xi_c,\xi_a)$ and $\min_{\xi_a\in\Xi_{a,\xi_c}} w_2(\xi_c,\xi_a) = w_2(\xi_c,\xi_a^*(\xi_c)) = w_1(\xi_c)$, where

$$\xi_b^*(\xi_c,\xi_a) = \arg\min_{\xi_b\in\Xi_{b,\xi_c,\xi_a}} w(\xi_c,\xi_a,\xi_b) \quad \text{and} \quad \xi_a^*(\xi_c) = \arg\min_{\xi_a\in\Xi_{a,\xi_c}} w_2(\xi_c,\xi_a).$$

For $a>b$, denote $\min_{\xi_a\in\Xi_{a,\xi_c,\xi_b}} w(\xi_c,\xi_a,\xi_b) = w(\xi_c,\xi_a^*(\xi_c,\xi_b),\xi_b) = w_2(\xi_c,\xi_b)$ and $\min_{\xi_b\in\Xi_{b,\xi_c}} w_2(\xi_c,\xi_b) = w_2(\xi_c,\xi_b^*(\xi_c)) = w_1(\xi_c)$, where

$$\xi_a^*(\xi_c,\xi_b) = \arg\min_{\xi_a\in\Xi_{a,\xi_c,\xi_b}} w(\xi_c,\xi_a,\xi_b) \quad \text{and} \quad \xi_b^*(\xi_c) = \arg\min_{\xi_b\in\Xi_{b,\xi_c}} w_2(\xi_c,\xi_b).$$

Moreover, denote $\min_{\xi_c\in\Xi_c} w_1(\xi_c) = w_1(\xi_c^*)$, where $\xi_c^* = \arg\min_{\xi_c\in\Xi_c} w_1(\xi_c)$. In addition, for $a<b$, denote $\xi_a^* = \xi_a^*(\xi_c^*)$ and $\xi_b^* = \xi_b^*(\xi_c^*,\xi_a^*)$. For $a>b$, denote $\xi_b^* = \xi_b^*(\xi_c^*)$ and $\xi_a^* = \xi_a^*(\xi_c^*,\xi_b^*)$.

Lemma 4.3.7 *The three-step minimization problem (4.35) is well-defined in the sense that the minimizer for each step exists. In particular, the minimizers of Problem (4.35) can be expressed as follows.*

(a) If $a < b$ and $0 \le \lambda < \frac{1}{2}$, then

$$\xi_c^* = \xi_c^m \vee \left(v_\theta \wedge \xi_c^M\right), \quad \xi_a^* = \sup\left\{\xi_a \in \Xi_{a,\xi_c^*} : \, A_{\xi_c^*}(\xi_a) < \xi_c^* + L_3\right\}, \quad \textit{and} \;\; \xi_b^* = \xi_b^m(\xi_c^*, \xi_a^*).$$

(b) If $a < b$ and $\frac{1}{2} < \lambda \le 1$, then

$$\xi_c^* = \xi_{L_3,h_2} \vee \xi_{L_3,h_3}, \quad \xi_a^* = \xi_c^* + a - c, \quad \textit{and} \quad \xi_b^* = \xi_b^m(\xi_c^*, \xi_a^*)$$

where

$$\begin{aligned}\xi_{L_3,h_2} &= \sup\left\{\xi_c \in [0, c - v_\theta] : \, h_2(\xi_c) \ge L_3\right\}, \\ \xi_{L_3,h_3} &= \sup\left\{\xi_c \in [0, v_\theta] : \, h_3(\xi_c) \le L_3\right\}.\end{aligned}$$

(c) If $a > b$ and $0 \le \lambda < \frac{1}{2}$, then

$$\xi_c^* = \xi_c^m \vee \left(v_\theta \wedge \xi_c^M\right), \quad \xi_b^* = \xi_b^m(\xi_c^*), \quad \textit{and} \quad \xi_a^* = \xi_b^* + a - b.$$

(d) If $a > b$ and $\frac{1}{2} < \lambda \le 1$, then

$$\xi_c^* = \xi_c^m \vee \left[(c - v_\theta) \wedge \xi_c^M\right], \quad \xi_b^* = \xi_c^* + b - c, \quad \textit{and} \quad \xi_a^* = \xi_c^* + a - c.$$

Theorem 4.3.8 *A contract I^* of the form*

$$\begin{aligned}I^*(x) = &(x - d_1)^+ - (x - d_1 - \xi_c^*)^+ + (x - d_2)^+ - (x - (d_2 + \xi_a^* \wedge \xi_b^* - \xi_c^*))^+ \\ &+ (x - d_3)^+ - (x - (d_3 + |\xi_b^* - \xi_a^*|))^+ + (x - d_4)^+\end{aligned} \tag{4.36}$$

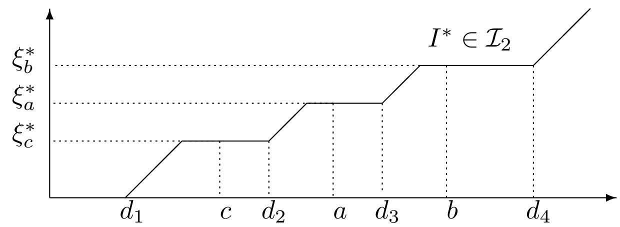
for some $(d_1, d_2, d_3, d_4) \in [0,\, c - \xi_c^*] \times [c,\, a \wedge b - \xi_a^* \wedge \xi_b^* + \xi_c^*] \times [a \wedge b,\, a \vee b - |\xi_a^* - \xi_b^*|] \times [a \vee b,\, \infty]$, satisfying $P_{I^*} = P_{\xi_c^*, \xi_a^*, \xi_b^*}$, is an optimal solution to Problem (4.23).

Remark 4.3.1 *Figure 4.2 illustrates the optimal form (4.36) in the case of $a < b$.*

By the proof of Theorem 4.3.8, we know that the optimal contract I^* in Theorem 4.3.8 satisfies $I^*(a) - I^*(b) = (a - b)^+$, and hence the optimal solution I^* in Theorem 4.3.8 is also the solution to Problem (4.23) when $\lambda = 1/2$. □

Next, we will derive the explicit expressions of the parameters in the optimal solution I^* given in Theorem 4.3.8 in the following four corollaries.

Figure 4.2: Optimal form of the contract when $a < b$



Corollary 4.3.9 *Suppose $a < b$ and $0 \leq \lambda < 1/2$ and let I^* be the optimal solution to Problem (4.23).*

(a) *In the case $h_2(v_\theta) \leq L_1 - c$:*

(i) *If $L_3 \leq h_4(v_\theta)$, then*

$$I^*(x) = (x - c + v_\theta)^+ - (x - c)^+ + (x - a + \xi_{a,0} - v_\theta)^+ - (x - a)^+,$$

where $\xi_{a,0}$ is the solution to the equation of $P_{I^*} = \xi_{a,0} + L_1 - a$.

(ii) *If $h_4(v_\theta) < L_3 \leq h_5(v_\theta)$, then*

$$I^*(x) = (x-d_1^*)^+ - (x-d_1^*-v_\theta)^+ + (x-d_2^*)^+ - (x-d_2^*-(a-L_1+L_3))^+ + (x-d_3^*)^+,$$

where $(d_1^*, d_2^*, d_3^*) \in [0,\, c-v_\theta] \times [c,\, L_1-L_3] \times [b,\, \infty]$ is the solution to the equation of $P_{I^*} = v_\theta + L_3$.

(iii) *If $h_5(v_\theta) < L_3$, then*

$$I^*(x) = x - (x - v_\theta)^+ + (x - c)^+ - (x - c - \xi_{a,1} + v_\theta)^+ + (x - b)^+,$$

where $\xi_{a,1}$ is the solution to the equation of $P_{I^*} = v_\theta + L_3$.

(b) *In the case $h_2(v_\theta) > L_1 - c$, then we have*

$$I^*(x) = (x - c + \xi_{L_1-c,\, h_2})^+ - (x - a)^+,$$

where $\xi_{L_1-c,\, h_2} = \inf\{\xi_c \in [v_\theta \wedge (c - v_\theta),\, v_\theta \vee (c - v_\theta)] : h_2(\xi_c) = L_1 - c\}$.

Corollary 4.3.10 *Suppose $a < b$ and $1/2 < \lambda \leq 1$ and let I^* be the optimal solution to Problem (4.23).*

(a) If $L_3 \leq h_2(0)$, then

$$I^*(x) = (x - c + \xi_{L_3,h_2})^+ - (x - a)^+,$$

where $\xi_{L_3,h_2} = \sup\{\xi_c \in [0, c - v_\theta] : h_2(\xi_c) \geq L_3\}$.

(b) If $h_2(0) < L_3 < h_3(0)$, then

$$I^*(x) = (x - c)^+ - (x - a)^+ + (x - d^*)^+,$$

where $d^* \in [b, \infty]$ satisifes $P_{I^*} = L_3$.

(c) If $h_3(0) \leq L_3$, then

$$I^*(x) = x - (x - \xi_{L_3,h_3})^+ + (x - c)^+ - (x - (c + \xi_b^* - \xi_{L_3,h_3}))^+ + (x - b)^+,$$

where $\xi_{L_3,h_3} = \sup\{\xi_c \in [0, v_\theta] : h_3(\xi_c) \leq L_3\}$ and $\xi_b^* \in [\xi_{L_3,h_3} + a - c, \xi_{L_3,h_3} + b - c]$ satisfies $P_{I^*} = \xi_{L_3,h_3} + L_3$.

Corollary 4.3.11 *Suppose $a > b$ and $0 \leq \lambda < 1/2$ and let I^* be an optimal solution to Problem (4.23).*

(a) In the case $h_2(v_\theta) \leq L_1 - c$:

(i) If $(b + L_3 - L_1)^+ + L_1 - b < h_6(v_\theta)$, then

$$I^*(x) = (x - c + v_\theta)^+ - (x - c)^+ + (x - (b - \xi_{b,0} + v_\theta))^+ - (x - a)^+,$$

where $\xi_{b,0} \in [v_\theta + (b + L_3 - L_1)^+, v_\theta + b - c]$ is the solution to the equation of $P_{I^*} = \xi_{b,0} - b + L_1$.

(ii) If $h_6(v_\theta) \leq (b + L_3 - L_1)^+ + L_1 - b < h_7(v_\theta)$, then

$$\begin{aligned} I^*(x) = &(x - d_1^*)^+ - (x - d_1^* - v_\theta)^+ + (x - c)^+ \\ &- (x - c - (b + L_3 - L_1)^+)^+ + (x - b)^+ - (x - a)^+ + (x - d_2^*)^+, \end{aligned}$$

where $(d_1^*, d_2^*) \in [0, c - v_\theta] \times [a, \infty]$ is the solution to the equation of $P_{I^*} = v_\theta + L_3 \vee (L_1 - b)$.

(iii) *If* $L_3 < h_7(v_\theta) \le (b + L_3 - L_1)^+ + L_1 - b$*, then*

$$I^*(x) = x - (x - v_\theta)^+ + (x - c)^+ - (x - c - (b + L_3 - L_1)^+)^+ + (x - b)^+.$$

(iv) *If* $h_7(v_\theta) \le L_3$*, then*

$$I^*(x) = x - (x - v_\theta)^+ + (x - c)^+ - (x - (c + \xi_{b,1} - v_\theta))^+ + (x - b)^+,$$

where $\xi_{b,1} \in [v_\theta + (b + L_3 - L_1)^+, v_\theta + b - c]$ *is the solution to the equation of* $P_{I^*} = v_\theta + L_3$*.*

(b) *In the case* $h_2(v_\theta) > L_1 - c$*, then we have* $I^*(x) = (x - c + \xi_{L_1-c,h_2})^+ - (x - a)^+$*, where* $\xi_{L_1-c,h_2} = \inf\{\xi_c \in [v_\theta \wedge (c - v_\theta), v_\theta \vee (c - v_\theta)] : h_2(\xi_c) = L_1 - c\}$*.*

Corollary 4.3.12 *Suppose* $a > b$ *and* $1/2 < \lambda \le 1$ *and let* I^* *be the optimal solution to Problem (4.23).*

(a) *If* $h_1(c - v_\theta) < L_3$*, then*

$$I^*(x) = x - (x - \xi_{L_3,h_1})^+ + (x - c)^+,$$

where $\xi_{L_3,h_1} = \sup\{\xi_c \in [v_\theta \wedge (c - v_\theta), v_\theta \vee (c - v_\theta)] : h_1(\xi_c) = L_3\}$*.*

(b) *If* $L_3 \le h_1(c - v_\theta)$*, then*

$$I^*(x) = (x - d_1^*)^+ - (x - d_1^* - c + v_\theta)^+ + (x - c)^+ - (x - a)^+ + (x - d_2^*)^+,$$

where $(d_1^*, d_2^*) \in [0, v_\theta] \times [a, \infty]$ *is the solution to the equation of* $P_{I^*} = c - v_\theta + L_3 \vee h_2(c - v_\theta)$*.*

4.4 Appendix

Proof of Proposition 4.2.1. We only prove (a) and (d). Other results of Proposition 4.2.1 can be proved similarly and are omitted.

(a) It is easy to see that $g_1(\xi_a)$ is continuous in $\xi_a \in [0, a]$. Since $\alpha < 1/(1+\theta)$, we have that $g_1'(\xi_a) = 1 - (1+\theta)S_X(a - \xi_a)$ is non-negative for $\xi_a \in [0, a - v_\theta)$ and is negative for $\xi_a \in (a - v_\theta, a]$. Hence, the desired results hold.

(d) Suppose $a < b$, note that $g_2(\xi_a) = \xi_a - P_{I^M_{\xi_a,\xi_a}} < \xi_a - P_{I^m_{\xi_a,\xi_a}} = g_1(\xi_a)$ for any $\xi_a \in [0, a]$. For each $(\xi_a, \xi_b) \in \Xi_{a,b}$, $\xi_a \le \xi_b$ by (4.13) and it is obvious that $P_{I^M_{\xi_a,\xi_b}}$ and $P_{I^m_{\xi_a,\xi_b}}$ are continuous and strictly increasing in $\xi_b \in [0, b]$. For any (ξ_a, ξ_1) and $(\xi_a, \xi_2) \in \Xi_{a,b}$ with $\xi_1 < \xi_2$, we have $0 \le P_{I^M_{\xi_a,\xi_2}} - P_{I^M_{\xi_a,\xi_1}} = (1+\theta)\int_{a+\xi_1-\xi_a}^{a+\xi_2-\xi_a} S_X(x)\mathrm{d}x \le (1+\theta)(\xi_2 - \xi_1)S_X(a) \le (1+\theta)\alpha\,(\xi_2 - \xi_1) < \xi_2 - \xi_1$, where the third inequality follows from $S_X(x) \le \alpha$ for any $x \ge \mathrm{VaR}_\alpha(X) = a$. Therefore, $\xi_b - P_{I^M_{\xi_a,\xi_b}}$ is continuous and strictly increasing in $\xi_b \in [0, b]$. ■

Proof of Lemma 4.2.2. We assume $a < b$. The proof for the case of $a > b$ is similar to the case of $a < b$ and is omitted.

(i) $\Rightarrow$ (ii). Suppose (4.6) holds, namely $g_1(a - v_\theta) \ge a - L_1$. Since $g_1(0) = 0 \le a - L_1$ and g_1 is continuous and increasing on $[0, a - v_\theta]$, there exists $\xi_a \in [0, a - v_\theta]$ such that $g_1(\xi_a) = a - L_1$, and moreover, $g_1(\xi_a) = a - L_1 \le L_2$. Consider the contract $I(x) = (x - a + \xi_a)^+ - (x - a)^+ \in \mathcal{I}$, it is easy to check that $I(a) = I(b) = \xi_a$ and $P_I = (1+\theta)\mathbb{E}[I(X)] = \xi_a - g_1(\xi_a) = \xi_a - a + L_1$. This contract I is acceptable, namely $I \in \mathcal{I}_1$, because the contract I satisfies $a - I(a) + P_I = a - \xi_a + \xi_a - a + L_1 = L_1$, and $I(b) - P_I = \xi_a - (\xi_a - a + L_1) = a - L_1 \le L_2$. Thus, $\mathcal{I}_1 \ne \emptyset$.

Meanwhile, by Proposition 4.2.1(b) and (d), we know that $g_1(\xi_a) = a - L_1 \le L_2$ implies that $g_2(v_\theta) \le g_2(\xi_a) < g_1(\xi_a) \le L_2$, namely (4.19) holds. Thus, (4.6) implies (4.19).

(ii) $\Rightarrow$ (iii). Suppose $\mathcal{I}_1 \ne \emptyset$. For any $I \in \mathcal{I}_1$, denote $\xi_a = I(a)$ and $\xi_b = I(b)$. We are going to check that (ξ_a, ξ_b) satisfies (4.13), (4.14), and (4.15). Since $I \in \mathcal{I}_1$, we have

$$\xi_b - L_2 \le P_I \le \xi_a + L_1 - a. \tag{4.37}$$

Furthermore, the 1-Lipschitz property of I implies $\xi_a \le \xi_b \le \xi_a + b - a$. Hence, (4.13) holds. Moreover, it is easy to see that $I^m_{\xi_a,\xi_b}(x) \le I(x) \le I^M_{\xi_a,\xi_b}(x)$ for all $x \ge 0$, and thus

$$P_{I^m_{\xi_a,\xi_b}} \le P_I \le P_{I^M_{\xi_a,\xi_b}}. \tag{4.38}$$

From (4.37) and (4.38), we have $\xi_b - L_2 \le P_{I^M_{\xi_a,\xi_b}}$ and $P_{I^m_{\xi_a,\xi_b}} \le \xi_a + L_1 - a$, namely (4.14) and (4.15) hold. Therefore, $(\xi_a, \xi_b) \in \Xi_{a,b}$ and thus $\Xi_{a,b} \neq \emptyset$.

(iii) $\Rightarrow$ (i). Suppose $\Xi_{a,b} \neq \emptyset$. For any $(\xi_a, \xi_b) \in \Xi_{a,b}$, we have

$$
\begin{aligned}
a - L_1 &\le \xi_a - (1+\theta)\left(\int_{a-\xi_a}^{a} + \int_{b-\xi_b+\xi_a}^{b}\right) S_X(x)\mathrm{d}x \le \xi_a - (1+\theta)\int_{a-\xi_a}^{a} S_X(x)\mathrm{d}x = g_1(\xi_a) \\
&\le a - v_\theta - (1+\theta)\int_{v_\theta}^{a} S_X(x)\mathrm{d}x = g_1(a - v_\theta),
\end{aligned}
$$

where the first inequality is from (4.15) and the last one is due to the fact that g_1 is increasing on $[0, a - v_\theta]$. Thus, (4.6) holds. ■

Proof of Lemma 4.2.3. We assume $a < b$. The proof for the case of $a > b$ is similar to the case of $a < b$ and is omitted.

For each $(\xi_a, \xi_b) \in \Xi_{a,b}$, by (4.13), we have $\xi_b - L_2 \le \xi_a + b \wedge (L_1 + L_2) - L_2 - a \le \xi_a + L_1 - a$, which, together with (4.14), implies $\xi_b - L_2 \le (L_1 - a + \xi_a) \wedge P_{I^M_{\xi_a,\xi_b}}$. Hence, by (4.15) and $P_{I^m_{\xi_a,\xi_b}} \le P_{I^M_{\xi_a,\xi_b}}$, we have $(\xi_b - L_2) \vee P_{I^m_{\xi_a,\xi_b}} \le (L_1 - a + \xi_a) \wedge P_{I^M_{\xi_a,\xi_b}}$. Therefore, by the definition of P_{ξ_a,ξ_b} given in (4.21), we have

$$
(\xi_b - L_2) \vee P_{I^m_{\xi_a,\xi_b}} \le P_{\xi_a,\xi_b} \le (L_1 - a + \xi_a) \wedge P_{I^M_{\xi_a,\xi_b}}. \tag{4.39}
$$

It is easy to check that any contract with the form of

$$
I(x) = (x - d_1)^+ - (x - d_1 - \xi_a)^+ + (x - d_2)^+ - (x - d_2 - \xi_b + \xi_a)^+ + (x - d_3)^+, \tag{4.40}
$$

for some $(d_1, d_2, d_3) \in [0,\, a - \xi_a] \times [a,\, b - \xi_b + \xi_a] \times [b,\, \infty]$, satisfies $I \in \mathcal{I}$, $I(a) = \xi_a$, $I(b) = \xi_b$, and $I^m_{\xi_a,\xi_b}(x) \le I(x) \le I^M_{\xi_a,\xi_b}(x)$ for all $x \ge 0$. Thus, $P_{I^m_{\xi_a,\xi_b}} \le P_I \le P_{I^M_{\xi_a,\xi_b}}$. In particular, when $d_1 = a - \xi_a$, $d_2 = b - \xi_b + \xi_a$, and $d_3 = \infty$, the form (4.40) is reduced to $I^m_{\xi_a,\xi_b}$. When $d_1 = 0$, $d_2 = a$, and $d_3 = b$, the form (4.40) is reduced to $I^M_{\xi_a,\xi_b}$. For the contract I of the form (4.40), its premium

$$
P_I = (1+\theta)\,\mathbb{E}[I(X)] = (1+\theta)\left(\int_{d_1}^{d_1+\xi_a} + \int_{d_2}^{d_2+\xi_b-\xi_a} + \int_{d_3}^{\infty}\right) S_X(x)\mathrm{d}x
$$

can be viewed as a function of (d_1, d_2, d_3). Obviously, the premium $P_I = P_I(d_1, d_2, d_3)$ is a real-valued continuous function on $[0, a - \xi_a] \times [a, b - \xi_b + \xi_a] \times [b, \infty]$. Since $[0, a - \xi_a] \times [a, b - \xi_b + \xi_a] \times [b, \infty]$ is a connected set, the image of $P_I(d_1, d_2, d_3)$ is also a connected set. Thus,

$$
\{P_I = (1+\theta)\,\mathbb{E}[I(X)] : I \text{ has the expression (4.40)}\} = \left[P_{I^m_{\xi_a,\xi_b}},\, P_{I^M_{\xi_a,\xi_b}}\right].
$$

For each $(\xi_a, \xi_b) \in \Xi_{a,b}$, note that $P_{\xi_a,\xi_b} \in \left[P_{I^m_{\xi_a,\xi_b}},\, P_{I^M_{\xi_a,\xi_b}}\right]$, thus there exists $I \in \mathcal{I}$ with the expression (4.40) such that $P_I = P_{\xi_a,\xi_b}$, and moreover, such $I \in \mathcal{I}_1$ due to (4.39).

The existence of the minimizer (ξ_a^*, ξ_b^*) of Problem (4.20) will be demonstrated in the proof of Theorems 4.2.4 and 4.2.5. Since $(\xi_a^*, \xi_b^*) \in \Xi_{a,b}$, by the above arguments, there exists $I^* \in \mathcal{I}_1$ of the form (4.22) such that $I^*(a) = \xi_a^*$, $I^*(b) = \xi_b^*$, and $P_{I^*} = P_{\xi_a^*,\xi_b^*}$. It can be easily checked that $V(I^*) = v(\xi_a^*, \xi_b^*)$. Meanwhile, for any $I \in \mathcal{I}_1$, we have $(I(a), I(b)) \in \Xi_{a,b}$ by the proof of Lemma 4.2.2 for (ii) $\Rightarrow$ (iii). From (4.21), we have $P_I \leq P_{I(a),\,I(b)}$ when $0 \leq \lambda < 1/2$, and $P_I \geq P_{I(a),\,I(b)}$ when $1/2 < \lambda \leq 1$. Therefore, $(2\lambda - 1)P_I \geq (2\lambda - 1)P_{I(a),\,I(b)}$ and

$$V(I) = \lambda a + (2\lambda - 1)P_I - \lambda I(a) + (1-\lambda)I(b) \geq v(I(a), I(b)) \geq \min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a, \xi_b), \tag{4.41}$$

which implies that $\min_{I\in\mathcal{I}_1} V(I) \geq \min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a,\xi_b) = v(\xi_a^*,\xi_b^*) = V(I^*) \geq \min_{I\in\mathcal{I}_1} V(I)$. Hence, $\min_{I\in\mathcal{I}_1} V(I) = V(I^*)$ and I^* is the optimal solution to Problem (4.12). Therefore, a contract I^* of the form (4.22) for some $(d_1, d_2, d_3) \in [0,\, a - \xi_a^*] \times [a,\, b - \xi_b^* + \xi_a^*] \times [b,\, \infty]$, satisfying $I^*(a) = \xi_a^*$, $I^*(b) = \xi_b^*$, $P_{I^*} = P_{\xi_a^*,\xi_b^*}$, is the optimal solution to Problem (4.12). ∎

Proof of Theorem 4.2.4. Assume $a < b$. For each $(\xi_a, \xi_b) \in \Xi_{a,b}$, we have $\xi_a \leq \xi_b$ by (4.13), $\xi_b - P_{I^M_{\xi_a,\xi_b}} \leq L_2$ by (4.14), and $P_{I^m_{\xi_a,\xi_b}} \leq L_1 - a + \xi_a$ by (4.15). Since $\xi_b - P_{I^M_{\xi_a,\xi_b}}$ and $P_{I^m_{\xi_a,\xi_b}}$ are strictly increasing in $\xi_b \in [\xi_a, \xi_a + b - a]$ by Proposition 4.2.1(d), we have $\xi_a - P_{I^M_{\xi_a,\xi_a}} \leq \xi_b - P_{I^M_{\xi_a,\xi_b}} \leq L_2$ and $P_{I^m_{\xi_a,\xi_a}} \leq P_{I^m_{\xi_a,\xi_b}} \leq L_1 - a + \xi_a$. Thus, $(\xi_a, \xi_a) \in \Xi_{a,b}$. From (4.14) and (4.15), we know that $(\xi_a, \xi_a) \in \Xi_{a,b}$ is equivalent to

$$g_2(\xi_a) \leq L_2 \quad \text{and} \quad a - L_1 \leq g_1(\xi_a). \tag{4.42}$$

(a) Consider the case $0 \leq \lambda < \frac{1}{2}$. By Lemma 4.2.3, $\min_{I\in\mathcal{I}_1} V(I) = \min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a, \xi_b)$, where $v(\xi_a, \xi_b) = \lambda a + (2\lambda - 1)P_{\xi_a,\xi_b} - \lambda\xi_a + (1-\lambda)\xi_b$ and $P_{\xi_a,\xi_b} = (L_1 - a + \xi_a) \wedge P_{I^M_{\xi_a,\xi_b}}$. For each $(\xi_a, \xi_b) \in \Xi_{a,b}$, since $\xi_a \leq \xi_b$ and $\xi_a - P_{I^M_{\xi_a,\xi_a}} \leq \xi_b - P_{I^M_{\xi_a,\xi_b}}$, together with the definition of P_{ξ_a,ξ_b} given by (4.21) and the facts that $-(x \wedge y) = (-x) \vee (-y)$ and

$kz + k(x \vee y) = k[(z+x) \vee (z+y)]$ for $k > 0$, we have

$$\begin{aligned}
v(\xi_a, \xi_b) &= \lambda a - \lambda \xi_a + (1-\lambda)\xi_b - (1-2\lambda)\left[(L_1 - a + \xi_a) \wedge P_{I^M_{\xi_a,\xi_b}}\right] \\
&= \lambda a - \lambda \xi_a + \lambda \xi_b + (1-2\lambda)\left[(\xi_b - L_1 + a - \xi_a) \vee \left(\xi_b - P_{I^M_{\xi_a,\xi_b}}\right)\right] \\
&\geq \lambda a + (1-2\lambda)\left[(a - L_1) \vee \left(\xi_a - P_{I^M_{\xi_a,\xi_a}}\right)\right] \\
&= (1-\lambda)a - (1-2\lambda)L_1 + (1-2\lambda)\left[g_2(\xi_a) - (a - L_1)\right]^+ \;=\; v(\xi_a, \xi_a).
\end{aligned}$$

Hence, $\min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a,\xi_b) \geq \min_{(\xi_a,\xi_a)\in\Xi_{a,b}} v(\xi_a,\xi_a)$, and since $(\xi_a,\xi_a) \in \Xi_{a,b}$, we have

$$\begin{aligned}
\min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a,\xi_b) &= \min_{(\xi_a,\xi_a)\in\Xi_{a,b}} v(\xi_a,\xi_a) \\
&= (1-\lambda)a - (1-2\lambda)L_1 + (1-2\lambda) \min_{(\xi_a,\xi_a)\in\Xi_{a,b}} \left[g_2(\xi_a) - (a-L_1)\right]^+ \\
&= (1-\lambda)a - (1-2\lambda)L_1 + (1-2\lambda)\left[\min_{(\xi_a,\xi_a)\in\Xi_{a,b}} g_2(\xi_a) - (a-L_1)\right]^+.
\end{aligned}$$

Note that $P_{I^M_{\xi_a,\xi_a}} = \xi_a - g_2(\xi_a)$ and then

$$P_{\xi_a,\xi_a} = (\xi_a - a + L_1) \wedge P_{I^M_{\xi_a,\xi_a}} = \xi_a - (a - L_1) \vee g_2(\xi_a), \tag{4.43}$$

(i) If $g_1(v_\theta) \geq a - L_1$, note that $g_2(v_\theta) \leq L_2$ by (4.19), thus $\xi_a = v_\theta$ satisfies condition (4.42), namely $(v_\theta, v_\theta) \in \Xi_{a,b}$. In this case,

$$\min_{(\xi_a,\xi_a)\in\Xi_{a,b}} g_2(\xi_a) \geq \min_{\xi_a\in[0,a]} g_2(\xi_a) = g_2(v_\theta) \geq \min_{(\xi_a,\xi_a)\in\Xi_{a,b}} g_2(\xi_a),$$

where the equality holds due to Proposition 4.2.1(b). Therefore, $\min_{(\xi_a,\xi_a)\in\Xi_{a,b}} g_2(\xi_a) = g_2(v_\theta)$ and $(\xi_a^*, \xi_b^*) = (v_\theta, v_\theta)$. It implies that $P_{\xi_a^*,\xi_b^*} = P_{v_\theta,v_\theta} = v_\theta - (a - L_1) \vee g_2(v_\theta)$ from (4.43), and

$$\min_{(\xi_a,\xi_a)\in\Xi_{a,b}} v(\xi_a,\xi_a) = v(v_\theta, v_\theta) = (1-\lambda)a - (1-2\lambda)L_1 + (1-2\lambda)\left[g_2(v_\theta) - (a-L_1)\right]^+.$$

By Lemma 4.2.3, a contract I^* of the form (4.22) satisfying $I^*(a) = v_\theta$, $I^*(b) = v_\theta$, and $P_{I^*} = P_{v_\theta,v_\theta}$, is the optimal solution to Problem (4.12). Note that $\xi_a^* = \xi_b^*$. Thus, $I^*(x) = (x - d_1)^+ - (x - d_1 - v_\theta)^+ + (x - d_3)^+$ for some $d_1 \in [0, a - v_\theta]$ and $d_3 \in [b, \infty]$ such that $P_{I^*} = v_\theta - (a - L_1) \vee g_2(v_\theta)$ is the optimal solution to Problem (4.12).

(ii) If $g_1(v_\theta) < a - L_1$, note that $g_1(a - v_\theta) \geq a - L_1$ from (4.6), thus there exists $\xi_1 \in [v_\theta \wedge (a - v_\theta),\, v_\theta \vee (a - v_\theta)]$ such that $g_1(\xi_1) = a - L_1$ due to the continuity and monotonicity of g_1 on this interval. From Proposition 4.2.1(d), we know $g_2(\xi) < g_1(\xi)$ for any $\xi \in [0, a]$. In particular, $g_2(\xi_1) < g_1(\xi_1) = a - L_1 \le L_2$ and thus ξ_1 satisfies condition (4.42), namely $(\xi_1, \xi_1) \in \Xi_{a,b}$. For any $(\xi_a, \xi_a) \in \Xi_{a,b}$, we have $[g_2(\xi_1) - (a - L_1)]^+ = 0 \le [g_2(\xi_a) - (a - L_1)]^+$. Then,

$$\left[\min_{(\xi_a,\xi_a)\in\Xi_{a,b}} g_2(\xi_a) - (a - L_1)\right]^+ = 0 = [g_2(\xi_1) - (a - L_1)]^+ ,$$

and $\xi_a^* = \xi_1$. In this case, we have $P_{\xi_a^*,\xi_b^*} = P_{\xi_1,\xi_1} = \xi_1 - (a - L_1) \vee g_2(\xi_1) = \xi_1 - a + L_1$ and $\min_{(\xi_a,\xi_a)\in\Xi_{a,b}} v(\xi_a, \xi_a) = v(\xi_1, \xi_1) = (1 - \lambda)a - (1 - 2\lambda)L_1$. Therefore, the optimal contract of the form (4.22) is reduced to $I^*(x) = (x-a+\xi_1)^+ - (x-a)^+$ with $d_1 = a-\xi_a$ and $d_3 = \infty$ because the contract I^* satisfies $I^*(a) = I^*(b) = \xi_1$ and $P_{I^*} = \xi_1 - g_1(\xi_1) = \xi_1 - a + L_1 = P_{\xi_1,\xi_1}$.

(b) For the case $\frac{1}{2} < \lambda \le 1$. By Lemma 4.2.3, we have $\min_{I\in\mathcal{I}_1} V(I) = \min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a, \xi_b)$, where $v(\xi_a, \xi_b) = \lambda a + (2\lambda - 1)P_{\xi_a,\xi_b} - \lambda\xi_a + (1 - \lambda)\xi_b$ and $P_{\xi_a,\xi_b} = (\xi_b - L_2) \vee P_{I^m_{\xi_a,\xi_b}}$. For each $(\xi_a, \xi_b) \in \Xi_{a,b}$, since $\xi_a \le \xi_b$ and $P_{I^m_{\xi_a,\xi_a}} \le P_{I^m_{\xi_a,\xi_b}}$, we have

$$\begin{aligned}
v(\xi_a, \xi_b) &= \lambda a - \lambda\xi_a + (1 - \lambda)\xi_b + (2\lambda - 1)\left[(\xi_b - L_2) \vee P_{I^m_{\xi_a,\xi_b}}\right] \\
&\ge \lambda a - \lambda\xi_a + (1 - \lambda)\xi_a + (2\lambda - 1)\left[(\xi_a - L_2) \vee P_{I^m_{\xi_a,\xi_a}}\right] \\
&= \ v(\xi_a, \xi_a) \ = \ \lambda a + (1 - 2\lambda)L_2 + (2\lambda - 1)\left[(\xi_a - L_2) \vee P_{I^m_{\xi_a,\xi_a}} - (\xi_a - L_2)\right] \\
&= \lambda a + (1 - 2\lambda)L_2 + (2\lambda - 1)\left[P_{I^m_{\xi_a,\xi_a}} - (\xi_a - L_2)\right]^+ \\
&= \lambda a + (1 - 2\lambda)L_2 + (2\lambda - 1)\left[L_2 - g_1(\xi_a)\right]^+ .
\end{aligned}$$

Hence, $\min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a, \xi_b) \ge \min_{(\xi_a,\xi_a)\in\Xi_{a,b}} v(\xi_a, \xi_a)$, and since $(\xi_a, \xi_a) \in \Xi_{a,b}$, we have

$$\begin{aligned}
\min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a, \xi_b) = \min_{(\xi_a,\xi_a)\in\Xi_{a,b}} v(\xi_a, \xi_a) \ &= \ \lambda a + (1 - 2\lambda)L_2 + (2\lambda - 1) \min_{(\xi_a,\xi_a)\in\Xi_{a,b}} [L_2 - g_1(\xi_a)]^+ \\
&= \ \lambda a + (1 - 2\lambda)L_2 + (2\lambda - 1)\left[L_2 - \max_{(\xi_a,\xi_a)\in\Xi_{a,b}} g_1(\xi_a)\right]^+ .
\end{aligned}$$

Note that $P_{I^m_{\xi_a,\xi_a}} = \xi_a - g_1(\xi_a)$ and then

$$P_{\xi_a,\xi_a} = (\xi_a - L_2) \vee P_{I^m_{\xi_a,\xi_a}} = \xi_a - L_2 \wedge g_1(\xi_a), \tag{4.44}$$

(i) If $g_2(a - v_\theta) \le L_2$, note that $a - L_1 \le g_1(a - v_\theta)$ by (4.6), thus $\xi_a = a - v_\theta$ satisfies condition (4.42), namely $(a - v_\theta,\ a - v_\theta) \in \Xi_{a,b} \subset [0, a] \times [0, b]$. In this case,

$$\max_{(\xi_a,\xi_a)\in\Xi_{a,b}} g_1(\xi_a) \le \max_{\xi_a\in[0,a]} g_1(\xi_a) = g_1(a - v_\theta) \le \max_{(\xi_a,\xi_a)\in\Xi_{a,b}} g_1(\xi_a),$$

where the equality holds due to Proposition 4.2.1(a). Therefore, $\max_{(\xi_a,\xi_a)\in\Xi_{a,b}} g_1(\xi_a) = g_1(a-v_\theta)$ and $(\xi_a^*,\xi_b^*) = (a-v_\theta,\, a-v_\theta)$. It implies that $P_{\xi_a^*,\xi_b^*} = P_{a-v_\theta,a-v_\theta} = a - v_\theta - L_2 \wedge g_1(a-v_\theta)$ due to (4.44), and $\min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a,\xi_a) = v(a-v_\theta, a-v_\theta) = \lambda a - (2\lambda - 1)\left[g_1(a-v_\theta)\wedge L_2\right]$.

By Lemma 4.2.3, a contract I^* of the form (4.22) satisfying $I^*(a) = a - v_\theta$, $I^*(b) = a - v_\theta$, and $P_{I^*} = P_{a-v_\theta,a-v_\theta}$, is the optimal solution to Problem (4.12). Note that $\xi_a^* = \xi_b^* = a - v_\theta$. Thus, $I^*(x) = (x-d_1)^+ - (x-d_1-a+v_\theta)^+ + (x-d_3)^+$ for any $d_1 \in [0, v_\theta]$ and $d_3 \in [b,\infty]$ such that $P_{I^*} = a - v_\theta - L_2 \wedge g_1(a-v_\theta)$ is the optimal solution to Problem (4.12).

(ii) If $g_2(a-v_\theta) > L_2$, note that $L_2 \geq g_2(v_\theta)$ by (4.19), thus there exists $\xi_2 \in [v_\theta \wedge (a-v_\theta),\, v_\theta \vee (a-v_\theta)]$ such that $L_2 = g_2(\xi_2)$ due to the continuity and monotonicity of g_2 as showed in Proposition 4.2.1(b). Moreover, $(\xi_2,\xi_2) \in \Xi_{a,b}$ from the observation $a - L_1 \leq L_2 = g_2(\xi_2) < g_1(\xi_2)$. For any $(\xi_a,\xi_a) \in \Xi_{a,b}$, we have $[L_2 - g_1(\xi_2)]^+ = 0 \leq [L_2 - g_1(\xi_a)]^+$. Thus,

$$\left[L_2 - \max_{(\xi_a,\xi_a)\in\Xi_{a,b}} g_2(\xi_a)\right]^+ = 0 = [L_2 - g_1(\xi_a)]^+,$$

and $\xi_a^* = \xi_2$. In this case, we have $P_{\xi_a^*,\xi_a^*} = P_{\xi_2,\xi_2} = \xi_2 - L_2 \wedge g_1(\xi_2) = \xi_2 - L_2$ due to (4.44), and $\min_{(\xi_a,\xi_a)\in\Xi_{a,b}} v(\xi_a,\xi_a) = v(\xi_2,\xi_2) = \lambda a + (1-2\lambda)L_2$. Therefore, the optimal contract of the form (4.22) is reduced to $I^*(x) = x - (x-\xi_2)^+ + (x-b)^+$ with $d_1 = 0$ and $d_3 = b$ because the contract I^* satisfies $I^*(a) = I^*(b) = \xi_2$ and $P_{I^*} = \xi_2 - g_2(\xi_2) = \xi_2 - L_2$. ■

Proof of Theorem 4.2.5. Assume $b < a$. For each $(\xi_a,\xi_b) \in \Xi_{a,b}$, we have $\xi_b \leq \xi_a \leq \xi_b + a - b$ by (4.16), $\xi_b - L_2 \leq P_{I^M_{\xi_a,\xi_b}}$ by (4.17), and $a - L_1 \leq \xi_a - P_{I^m_{\xi_a,\xi_b}}$ by (4.18). Since $P_{I^M_{\xi_a,\xi_b}}$ and $\xi_a - P_{I^m_{\xi_a,\xi_b}}$ are continuous and strictly increasing in $\xi_a \in [0,a]$ by Proposition 4.2.1(e), we have $\xi_b - L_2 \leq P_{I^M_{\xi_a,\xi_b}} \leq P_{I^M_{\xi_b+a-b,\xi_b}}$ and $a - L_1 \leq \xi_a - P_{I^m_{\xi_a,\xi_b}} \leq \xi_b + a - b - P_{I^m_{\xi_b+a-b,\xi_b}}$. Thus, $(\xi_b + a - b,\, \xi_b) \in \Xi_{a,b}$. By (4.17) and (4.18), we know that $(\xi_b + a - b, \xi_b) \in \Xi_{a,b}$ is equivalent to

$$g_2(\xi_b) \leq L_2 \quad\text{and}\quad g_3(\xi_b) \geq b - L_1. \tag{4.45}$$

(a) Consider the case $0 \leq \lambda < \frac{1}{2}$. By Lemma 4.2.3, $\min_{I\in\mathcal{I}_1} V(I) = \min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a,\xi_b)$, where $v(\xi_a,\xi_b) = \lambda a + (2\lambda-1)P_{\xi_a,\xi_b} - \lambda\xi_a + (1-\lambda)\xi_b$ and $P_{\xi_a,\xi_b} = (L_1 - a + \xi_a) \wedge P_{I^M_{\xi_a,\xi_b}}$.

For $(\xi_a, \xi_b) \in \Xi_{a,b}$, since $\xi_a \leq \xi_b + a - b$ and $P_{I^M_{\xi_a,\xi_b}} \leq P_{I^M_{\xi_b+a-b,\xi_b}}$, we have

$$
\begin{aligned}
v(\xi_a, \xi_b) &= \lambda a - \lambda\xi_a + (1-\lambda)\xi_b - (1-2\lambda)\left[(L_1 - a + \xi_a) \wedge P_{I^M_{\xi_a,\xi_b}}\right] \\
&\geq \lambda a - \lambda(\xi_b + a - b) + (1-\lambda)\xi_b - (1-2\lambda)\left[(L_1 - b + \xi_b) \wedge P_{I^M_{\xi_b+a-b,\xi_b}}\right] \\
&= v(\xi_b + a - b, \xi_b) \\
&= \lambda b + (1-2\lambda)(b - L_1) - (1-2\lambda)\left[(L_1 - b + \xi_b) \wedge P_{I^M_{\xi_b+a-b,\xi_b}} - (L_1 - b + \xi_b)\right] \\
&= (1-\lambda)b - (1-2\lambda)L_1 + (1-2\lambda)\left[L_1 - b + \xi_b - P_{I^M_{\xi_b+a-b,\xi_b}}\right]^+ \\
&= (1-\lambda)b - (1-2\lambda)L_1 + (1-2\lambda)\left[g_2(\xi_b) - (b - L_1)\right]^+ .
\end{aligned}
$$

Hence, $\min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a, \xi_b) = \min_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} v(\xi_b+a-b, \xi_b)$, and since $(\xi_b+a-b, \xi_b) \in \Xi_{a,b}$, we have

$$
\begin{aligned}
\min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a, \xi_b) &= \min_{(\xi_b+a-b,\xi_b)\in\Xi_{a,b}} v(\xi_b + a - b, \xi_b) \\
&= (1-\lambda)b - (1-2\lambda)L_1 + (1-2\lambda) \min_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} \left[g_2(\xi_b) - (b - L_1)\right]^+ \\
&= (1-\lambda)b - (1-2\lambda)L_1 + (1-2\lambda)\left[\min_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} g_2(\xi_b) - (b - L_1)\right]^+ .
\end{aligned}
$$

Note that $P_{I^M_{\xi_b+a-b,\xi_b}} = \xi_b - g_2(\xi_b)$ and then

$$
P_{\xi_b+a-b,\xi_b} = (\xi_b + a - b - a + L_1) \wedge P_{I^M_{\xi_b+a-b,\xi_b}} = \xi_b - (b - L_1) \vee g_2(\xi_b), \tag{4.46}
$$

(i) If $g_3(v_\theta) \geq b - L_1$, note that $g_2(v_\theta) \leq L_2$ by (4.19), thus $\xi_b = v_\theta$ satisfies condition (4.45), namely $(v_\theta + a - b, v_\theta) \in \Xi_{a,b}$. In this case,

$$
\min_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} g_2(\xi_b) \geq \min_{\xi_b\in[0,b]} g_2(\xi_b) = g_2(v_\theta) \geq \min_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} g_2(\xi_b),
$$

where the equality holds due to Proposition 4.2.1(b). Therefore, $\min_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} g_2(\xi_b) = g_2(v_\theta)$ and $(\xi_a^*, \xi_b^*) = (v_\theta+a-b, v_\theta)$. It implies that $P_{\xi_a^*,\xi_b^*} = P_{v_\theta+a-b,v_\theta} = v_\theta-(b-L_1)\vee g_2(v_\theta)$ due to (4.46), and

$$
\begin{aligned}
\min_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} v(\xi_b + a - b, \xi_b) &= v(v_\theta + a - b, v_\theta) \\
&= (1-\lambda)b - (1-2\lambda)L_1 + (1-2\lambda)\left[g_2(v_\theta) - (b - L_1)\right]^+ .
\end{aligned}
$$

By Lemma 4.2.3, a contract I^* of the form (4.22) satisfying $I^*(a) = v_\theta + a - b$, $I^*(b) = v_\theta$ and $P_{I^*} = P_{v_\theta+a-b,v_\theta}$, is the optimal solution to Problem (4.12). In this case, note that $\xi_a^* = v_\theta + a - b$ and $\xi_b^* = v_\theta$. It implies that the range for d_2 given in (4.22) is reduced to a single point set, that is $d_2 \in [b,\ a - I^*(a) + I^*(b)] = \{b\}$ and then, $d_2 = b$. Hence, the optimal solution to Problem (4.12) is reduced to $I^*(x) = (x-d_1) - (x-d_1-v_\theta)^+ + (x-b)^+ - (x-a)^+ + (x-d_3)^+$ for some $d_1 \in [0, b-v_\theta]$ and $d_3 \in [a,\infty]$ such that $P_{I^*} = v_\theta - (b-L_1) \vee g_2(v_\theta)$.

(ii) If $g_3(v_\theta) < b - L_1$, note that $g_3(b - v_\theta) \ge b - L_1$ by (4.6) and g_3 is continuous on $[0, b]$, thus there exists $\xi_3 \in [v_\theta \wedge (b - v_\theta),\ v_\theta \vee (b - v_\theta)]$ such that $g_3(\xi_3) = b - L_1$. From Proposition 4.2.1(e), we known that $g_2(\xi) < g_3(\xi)$ for all $\xi \in [0, b]$. In particular, $g_2(\xi_3) < g_3(\xi_3) = b - L_1 \le L_2$ and then ξ_3 satisfies condition (4.45), namely $(\xi_3 + a - b, \xi_3) \in \Xi_{a,b}$. For any $(\xi_b + a - c, \xi_b) \in \Xi_{a,b}$, we have $[g_2(\xi_3) - (b - L_1)]^+ = 0 \le [g_2(\xi_b) - (b - L_1)]^+$. Then,

$$\left[\min_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} g_2(\xi_b) - (b-L_1)\right]^+ = 0 = [g_2(\xi_3) - (b-L_1)]^+ ,$$

and $\xi_b^* = \xi_3$. In this case, we have $P_{\xi_a^*,\xi_b^*} = P_{\xi_3+a-b,\xi_3} = \xi_3 - (b - L_1) \vee g_2(\xi_3) = \xi_3 - b + L_1$ due to (4.46) and

$$\min_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} v(\xi_b + a - b, \xi_b) = v(\xi_3 + a - b, \xi_3) = (1-\lambda)b - (1-2\lambda)L_1.$$

The optimal contract of the form (4.22) is reduced to $I^*(x) = (x - b + \xi_3)^+ - (x - a)^+$ with $d_1 = b - \xi_3$, $d_2 = b$, and $d_3 = \infty$ because the contract I^* satisfies $I^*(a) = \xi_3 + a - b$, $I^*(b) = \xi_3$ and $P_{I^*} = \xi_3 - g_3(\xi_3) = \xi_3 - b + L_1$.

(b) Consider the case $\frac{1}{2} < \lambda \le 1$. By Lemma 4.2.3, $\min_{I\in\mathcal{I}_1} V(I) = \min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a, \xi_b)$, where $v(\xi_a, \xi_b) = \lambda a - \lambda\xi_a + (1-\lambda)\xi_b + (2\lambda-1)P_{\xi_a,\xi_b}$ and $P_{\xi_a,\xi_b} = (\xi_b - L_2) \vee P_{I^m_{\xi_a,\xi_b}}$. For any $(\xi_a, \xi_b) \in \Xi_{a,b}$, it is easy to check that $\xi_a \le \xi_b + a - b$ and $P_{I^m_{\xi_a,\xi_b}} - \xi_a \ge P_{I^m_{\xi_b+a-b,\xi_b}} - (\xi_b + a - b)$, thus

$$\begin{aligned}
v(\xi_a,\xi_b) &= \lambda a + (1-\lambda)(\xi_b - \xi_a) + (2\lambda-1)\left[(\xi_b - L_2 - \xi_a) \vee \left(P_{I^m_{\xi_a,\xi_b}} - \xi_a\right)\right] \\
&\ge \lambda a + (1-\lambda)(b-a) + (2\lambda-1)\left[(b - a - L_2) \vee \left(P_{I^m_{\xi_b+a-b,\xi_b}} - (\xi_b + a - b)\right)\right] \\
&= v(\xi_b + a - b, \xi_b) \ = \ \lambda b + (2\lambda-1)\left[(-L_2) \vee \left(P_{I^m_{\xi_b+a-b,\xi_b}} - \xi_b\right)\right] \\
&= \lambda b + (1-2\lambda)L_2 + (2\lambda-1)\left[P_{I^m_{\xi_b+a-b,\xi_b}} - (\xi_b - L_2)\right]^+ \\
&= \lambda b + (1-2\lambda)L_2 + (2\lambda-1)\left[L_2 - g_3(\xi_b)\right]^+ .
\end{aligned}$$

Hence, $\min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a,\xi_b) \geq \min_{(\xi_b+a-b,\xi_b)\in\Xi_{a,b}} v(\xi_b+a-b,\xi_b)$, and since $(\xi_b+a-b,\xi_b)\in\Xi_{a,b}$, we have

$$\begin{aligned}\min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a,\xi_b) &= \min_{(\xi_b+a-b,\xi_b)\in\Xi_{a,b}} v(\xi_b+a-b,\xi_b)\\ &= \lambda b + (1-2\lambda)L_2 + (2\lambda-1)\min_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} [L_2 - g_3(\xi_b)]^+\\ &= \lambda b + (1-2\lambda)L_2 + (2\lambda-1)\left[L_2 - \max_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} g_3(\xi_b)\right]^+.\end{aligned}$$

Note that $P_{I^m_{\xi_b+a-b,\xi_b}} = \xi_b - g_3(\xi_b)$ and then

$$P_{\xi_b+a-b,\xi_b} = (\xi_b - L_2)\vee P_{I^m_{\xi_b+a-b,\xi_b}} = \xi_b - L_2\wedge g_3(\xi_b), \tag{4.47}$$

(i) If $g_2(b-v_\theta)\leq L_2$, note that $g_3(b-v_\theta)\geq b-L_1$ by (4.6), thus $\xi_b = b-v_\theta$ satisfies condition (4.45), namely $(a-v_\theta, b-v_\theta)\in\Xi_{a,b}$. It implies that

$$\max_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} g_3(\xi_b) \leq \max_{\xi_b\in[0,b]} g_3(\xi_b) = g_3(b-v_\theta) \leq \max_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} g_3(\xi_b).$$

where the equality holds due to Proposition 4.2.1(c). Therefore, we obtain that $\max_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} g_3(\xi_b) = g_3(b-v_\theta)$ and $(\xi_a^*,\xi_b^*) = (a-v_\theta, b-v_\theta)$. It implies that $P_{\xi_a^*,\xi_b^*} = P_{a-v_\theta,b-v_\theta} = b - v_\theta - g_3(b-v_\theta)\wedge L_2$ due to (4.47), and

$$\min_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} v(\xi_b+a-b,\xi_b) = \lambda b + (1-2\lambda)L_2 + (2\lambda-1)\left[L_2 - g_3(b-v_\theta)\right]^+.$$

By Lemma 4.2.3, a contract I^* of the form (4.22) satisfying $I^*(a) = a-v_\theta$, $I^*(b) = b-v_\theta$, and $P_{I^*} = P_{a-v_\theta,b-v_\theta}$, is the optimal solution to Problem (4.12). Note that in this case, $\xi_a^* = a-v_\theta$ and $\xi_b^* = b-v_\theta$. Hence, $d_2\in[b, a-I^*(a)+I^*(b)] = \{b\}$ and thus $d_2 = b$. Therefore, the optimal solution I^* is reduced to $I^*(x) = (x-d_1)^+ - (x-d_1-b+v_\theta)^+ + (x-b)^+ - (x-a)^+ + (x-d_3)^+$ for some $d_1\in[0,v_\theta]$ and $d_3\in[a,\infty]$ such that $P_{I^*} = b - v_\theta - L_2\wedge g_3(b-v_\theta)$.

(ii) If $g_2(b-v_\theta) > L_2$, note that $g_2(v_\theta)\leq L_2$ due to (4.19), thus there exists $\xi_4\in[v_\theta\wedge(b-v_\theta),\, v_\theta\vee(b-v_\theta)]$ such that $g_2(\xi_4) = L_2$ due to the continuity and monotonicity of g_2. Since $a-L_1\leq L_2 = g_2(\xi_4) < g_3(\xi_4)$, we have that ξ_4 satisfies (4.45), namely $(\xi_4+a-b,\,\xi_4)\in\Xi_{a,b}$. For all $(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}$, we have $[L_2-g_3(\xi_4)]^+ = 0 \leq [L_2-g_3(\xi_b)]^+$. Thus,

$$\left[L_2 - \max_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} g_3(\xi_b)\right]^+ = 0 = \left[L_2 - g_3(\xi_4)\right]^+,$$

and $\xi_b^* = \xi_4$. In this case, we have $P_{\xi_a^*,\xi_b^*} = P_{\xi_4+a-b,\xi_4} = \xi_4 - L_2 \wedge g_3(\xi_4) = \xi_4 - L_2$ due to (4.47), and $\min_{(\xi_b+a-b,\xi_b)\in\Xi_{a,b}} v(\xi_b+a-b,\xi_b) = v(\xi_4+a-b,\xi_4) = \lambda b+(1-2\lambda)L_2$. Therefore, the optimal contract of the form (4.22) is reduced to $I^*(x) = x-(x-\xi_4)^+ + (x-b)^+$ because the contract f^* satisfies $I^*(a) = \xi_4 + a - b$, $I^*(b) = \xi_4$, and $P_{I^*} = \xi_4 - g_2(\xi_4) = \xi_4 - L_2 = P_{\xi_4+a-b,\xi_4}$. ■

Proof of Proposition 4.3.1. **(a)** Obviously, $h_1(\xi_c) = (1+\theta)\left(\int_0^{\xi_c} + \int_c^\infty\right) S_X(x)\mathrm{d}x - \xi_c$ is continuous and differentiable with $h_1'(\xi_c) = (1+\theta)S_X(\xi_c)-1$. Since $h_1'(\xi_c)$ is decreasing in ξ_c, we obtain that $h_1(\xi_c)$ is a concave function of ξ_c. For any $0 \le \xi_c < v_\theta$, we have $S_X(\xi_c) > \frac{1}{1+\theta}$, where $v_\theta = \mathrm{VaR}_{\frac{1}{1+\theta}}(X) = \inf\left\{x \ge 0 : S_X(x) \le \frac{1}{1+\theta}\right\}$. Thus, $h_1'(\xi_c) = (1+\theta)S_X(\xi_c)-1 > 0$ for any $0 \le \xi_c < v_\theta$, and $h_1(\xi_c)$ is strictly increasing on $[0, v_\theta)$. For any $c \ge \xi_c > v_\theta$, we have $S_X(\xi_c) \le \frac{1}{1+\theta}$. Thus, $h_1'(\xi_c) = (1+\theta)S_X(\xi_c) - 1 \le 0$ for any $c \ge \xi_c > v_\theta$, and $h_1(\xi_c)$ is decreasing on $(v_\theta, c]$. Hence, $\max_{\xi_c\in[0,c]} h_1(\xi_c) = h_1(v_\theta)$.

(b) Obviously, $h_2(\xi_c) = (1+\theta)\int_{c-\xi_c}^a S_X(x)\mathrm{d}x - \xi_c$ is continuous and differentiable with $h_2'(\xi_c) = (1+\theta)S_X(c-\xi_c) - 1$. For $\xi_c < c - v_\theta$, we have $c - \xi_c > v_\theta$ and $S_X(c-\xi_c) \le \frac{1}{1+\theta}$. For $\xi_c > c - v_\theta$, we have $c - \xi_c < v_\theta$ and $S_X(c-\xi_c) > \frac{1}{1+\theta}$. Thus, $h_2(\xi_c)$ is decreasing on $[0, c-v_\theta)$, strictly increasing on $(c-v_\theta, c]$ and $\min_{\xi_c\in[0,c]} h_2(\xi_c) = h_2(c-v_\theta)$. Since $c < a$ and $S_X(x)$ is continuous and decreasing in $x \ge 0$, we have, for $\xi_c \in [0,c]$,

$$\begin{aligned}
h_1(\xi_c) - h_2(\xi_c) &= (1+\theta)\left(\int_0^{\xi_c} + \int_c^a + \int_a^\infty\right) S_X(x)\mathrm{d}x - (1+\theta)\left(\int_{c-\xi_c}^c + \int_c^a\right) S_X(x)\mathrm{d}x \\
&= (1+\theta)\left(\int_0^{\xi_c} - \int_{c-\xi_c}^c + \int_a^\infty\right) S_X(x)\mathrm{d}x \\
&= (1+\theta)\int_0^{\xi_c} \left[S_X(x) - S_X(x+c-\xi_c)\right]\mathrm{d}x + (1+\theta)\int_a^\infty S_X(x)\mathrm{d}x > 0,
\end{aligned}$$

where $S_X(x) \ge S_X(x+c-\xi_c)$ and $S_X(a) = \alpha > 0$. ■

Proof of Proposition 4.3.2. We prove (b) for the function A_{ξ_c} only. The proofs for all the other functions and results in (a)-(f) can be obtained using similar arguments and are omitted.

(b) Clearly, $A_{\xi_c}(\xi_a) = P_{I^M_{\xi_c,\xi_a,\xi_a}} = (1+\theta)\left(\int_0^{\xi_c} + \int_c^{c+\xi_a-\xi_c} + \int_b^\infty\right) S_X(x)\mathrm{d}x$ is continuous and strictly increasing in ξ_a with $A_{\xi_c}'(\xi_a) = (1+\theta)S_X(c+\xi_a-\xi_c) > 0$. Note that $S_X(c+\xi_a-\xi_c) \le S_X(c) = 1-\gamma < \frac{1}{1+\theta}$ and $\frac{\mathrm{d}}{\mathrm{d}\xi_a}\left[\xi_a - A_{\xi_c}(\xi_a)\right] = 1-(1+\theta)S_X(c+\xi_a-\xi_c) > 0$. Thus, $\xi_a - A_{\xi_c}(\xi_a)$ is continuous and strictly increasing in $\xi_a \in [\xi_c, \xi_c+b-c]$. ■

Proof of Proposition 4.3.3. The proof of this proposition is similar to the proof of Propositions 4.3.1 and 4.3.2 and is omitted. ■

Proof of Proposition 4.3.4. (a) Note that $\Xi_c \subset [0,c]$. If $\Xi_c = [0,c]$, then $\xi_c^m = 0$, $\xi_c^M = c$ and the proof is done. Now, assume $\Xi_c \neq [0,c]$. From (4.27) and (4.28), $\xi_c \in \Xi_c$ is equivalent to $h_1(\xi_c) \geq L_3$ and $h_2(\xi_c) \leq L_1 - c$. From Proposition 4.3.1(a) and (b), we have that h_1 is concave and h_2 is convex on $[0,c]$. Denote $\xi_c^m = \inf \Xi_c$ and $\xi_c^M = \sup \Xi_c$. Then $0 \leq \xi_c^m \leq \xi_c^M \leq c$ because $\Xi_c \subset [0,c]$. There exists a sequence $\{x_n\}_{n=1}^{\infty} \subset \Xi_c$ such that $x_n \to \xi_c^m$ as $n \to \infty$. For each n, we have $h_1(x_n) \geq L_3$ and $h_2(x_n) \leq L_1 - c$ because $x_n \in \Xi_c$. By continuity of h_1 and h_2, $h_1(\xi_c^m) = \lim_{n\to\infty} h_1(x_n) \geq L_3$ and $h_2(\xi_c^m) = \lim_{n\to\infty} h_2(x_n) \leq L_1 - c$ and thus, $\xi_c^m \in \Xi_c$. Using a similar argument, we can prove $\xi_c^M \in \Xi_c$. For any $\xi_c \in (\xi_c^m, \xi_c^M)$, there exists $\delta \in (0,1)$ such that $\xi_c = \delta\xi_c^m + (1-\delta)\xi_c^M$. It is easy to see that $\xi_c \in \Xi_c$ because $h_1(\xi_c) = h_1(\delta\xi_c^m + (1-\delta)\xi_c^M) \geq \delta h_1(\xi_c^m) + (1-\delta)h_1(\xi_c^M) \geq \delta L_3 + (1-\delta)L_3 = L_3$ from the concavity of h_1; and $h_2(\xi_c) = h_2(\delta\xi_c^m + (1-\delta)\xi_c^M) \leq \delta h_2(\xi_c^m) + (1-\delta)h_2(\xi_c^M) \leq \delta(L_1 - c) + (1-\delta)(L_1 - c) = L_1 - c$ from the convexity of h_2. Therefore, $\Xi_c = [\xi_c^m, \xi_c^M] \subset [0,c]$.

The proofs of (b) and (c) are similar to (a) and are omitted. ∎

Proof of Lemma 4.3.5. We assume $a < b$. The proof for the case of $a > b$ is similar to the case of $a < b$ and is omitted.

(i) ⇒ (ii). Suppose (4.6) and (4.7) hold, which are equivalent to $h_2(c - v_\theta) \leq L_1 - c$ and $L_3 \leq h_1(v_\theta)$, respectively. We will prove $\mathcal{I}_2 \neq \emptyset$ by considering the following two cases.

Case 1: If $h_2(0) \vee h_2(c) \geq L_1 - c$, by the continuity of h_2 and (4.6), there exists $\xi_c \in [0,c]$ such that $h_2(\xi_c) = L_1 - c$, and thus $L_3 \leq L_1 - c = h_2(\xi_c) < h_1(\xi_c)$. Consider the contract $I(x) = (x - c + \xi_c)^+ - (x - a)^+ \in \mathcal{I}$. It is easy to check that $I(c) = \xi_c$, $I(a) = \xi_c + a - c$, and $P_I = h_2(\xi_c) + \xi_c = L_1 - c + \xi_c$. Thus $I \in \mathcal{I}_2$ since $a - I(a) + P_I = a - (\xi_c + a - c) + L_1 - c + \xi_c = L_1$ and $P_I - I(c) = L_1 - c \geq L_3$.

Case 2: If $h_2(0) \vee h_2(c) < L_1 - c$, then $h_2(\xi_c) \leq L_1 - c$ for all $\xi_c \in [0,c]$, and in particular, $h_2(v_\theta) \leq L_1 - c$. Note that $L_3 \leq h_1(v_\theta)$ by (4.7), we have

$$v_\theta - c + L_1 \geq h_2(v_\theta) + v_\theta = P_{I^m_{v_\theta,\, v_\theta+a-c}} \quad \text{and} \quad v_\theta + L_3 \leq h_1(v_\theta) + v_\theta = P_{I^M_{v_\theta,\, v_\theta+a-c}},$$

where $I^m_{v_\theta,\, v_\theta+a-c}(x) = (x - c + v_\theta)^+ - (x - a)^+$ and $I^M_{v_\theta,\, v_\theta+a-c}(x) = x - (x - v_\theta)^+ + (x - c)^+$ for all $x \geq 0$. Since $c \leq L_1 - L_3$, we have $v_\theta + L_3 \leq v_\theta - c + L_1$. Note that $P_{I^m_{v_\theta,v_\theta+a-c}} \leq P_{I^M_{v_\theta,v_\theta+a-c}}$, and thus $(v_\theta + L_3) \vee P_{I^m_{v_\theta,v_\theta+a-c}} \leq (v_\theta - c + L_1) \wedge P_{I^M_{v_\theta,v_\theta+a-c}}$. Using similar arguments to those used in the proof of Lemma 4.2.3, we know that as a function of $(d_1, d_2) \in [0, c - v_\theta] \times [a, \infty]$, $P_I = P_I(d_1, d_2) = (1+\theta)\left(\int_{d_1}^{d_1+v_\theta} + \int_c^{d_2}\right) S_X(x)\mathrm{d}x$ can take all its intermediate values in the interval $[P_{I^m_{v_\theta,\, v_\theta+a-c}}, P_{I^M_{v_\theta,\, v_\theta+a-c}}]$. Thus, there exists $(d_1, d_2) \in [0, c - v_\theta] \times [a, \infty]$ such that $P_I(d_1, d_2) = (v_\theta + L_3) \vee P_{I^m_{v_\theta,\, v_\theta+a-c}}$. Consider the contract $I(x) = (x-d_1)^+ - (x-d_1-v_\theta)^+ + (x-c)^+ - (x-d_2)^+$, it is easy to check that $I(c) = v_\theta$, $I(a) = v_\theta + a - c$, and $P_I = P_I(d_1, d_2) = (v_\theta + L_3) \vee P_{I^m_{v_\theta,\, v_\theta+a-c}}$. Thus, $I(c) + L_3 = v_\theta + L_3 \leq P_I \leq v_\theta - c + L_1 = I(a) - a + L_1$ and $I \in \mathcal{I}_2$.

Therefore, by combining **Cases 1 and 2**, we get $\mathcal{I}_2 \neq \emptyset$.

(ii) $\Rightarrow$ (iii). Suppose $\mathcal{I}_2 \neq \emptyset$. For any $I \in \mathcal{I}_2$, denote $\xi_c = I(c)$, $\xi_a = I(a)$ and $\xi_b = I(b)$. Note that for $a < b$ and $I \in \mathcal{I}_2$, we have $\xi_a \leq \xi_b$ and $\xi_c + L_3 \leq P_I \leq \xi_a - a + L_1$, and thus (4.24) holds. It is easy to check that $I^m_{\xi_c,\xi_a,\xi_b}(x) \leq I(x) \leq I^M_{\xi_c,\xi_a,\xi_b}(x)$ for all $x \geq 0$ and thus $P_{I^m_{\xi_c,\xi_a,\xi_b}} \leq P_I \leq P_{I^M_{\xi_c,\xi_a,\xi_b}}$. Moreover we get $(\xi_c + L_3) \vee P_{I^m_{\xi_c,\xi_a,\xi_b}} \leq P_I \leq (\xi_a - a + L_1) \wedge P_{I^M_{\xi_c,\xi_a,\xi_b}}$ and it implies that (4.25) and (4.26) hold for (ξ_c, ξ_a, ξ_b). By its definition, $(\xi_c, \xi_a, \xi_b) \in \Xi_{c,a,b}$ and then $\Xi_{c,a,b} \neq \emptyset$.

(iii) $\Rightarrow$ (i). Suppose $\Xi_{c,a,b} \neq \emptyset$. From (4.25), we get

$$
\begin{aligned}
L_3 &\leq (1+\theta)\left(\int_0^{\xi_c} + \int_c^{c+\xi_a-\xi_c} + \int_a^{a+\xi_b-\xi_a} + \int_b^{\infty}\right) S_X(x)\mathrm{d}x - \xi_c \\
&\leq (1+\theta)\left(\int_0^{\xi_c} + \int_c^{\infty}\right) S_X(x)\mathrm{d}x - \xi_c = h_1(\xi_c) \leq h_1(v_\theta).
\end{aligned}
$$

Thus, (4.7) holds. From (4.26) and the fact that $\xi_a - A^m_{\xi_c}(\xi_a)$ is increasing in ξ_a, we get

$$
\begin{aligned}
a - L_1 &\leq \xi_a - (1+\theta)\left(\int_{c-\xi_c}^{c} + \int_{a-\xi_a+\xi_c}^{a} + \int_{b-\xi_b+\xi_a}^{b}\right) S_X(x)\mathrm{d}x \\
&\leq \xi_a - (1+\theta)\left(\int_{c-\xi_c}^{c} + \int_{a-\xi_a+\xi_c}^{a}\right) S_X(x)\mathrm{d}x = \xi_a - A^m_{\xi_c}(\xi_a) \\
&\leq \xi_c + a - c - A^m_{\xi_c}(\xi_c + a - c) = a - c - h_2(\xi_c) \leq a - c - h_2(c - v_\theta),
\end{aligned}
$$

where $h_2(\xi_c) = A^m_{\xi_c}(\xi_c + a - c) - \xi_c$. Thus, (4.6) holds. ■

Proof of Lemma 4.3.6. We assume $a < b$. The proof for the case of $a > b$ is similar to the case of $a < b$ and is omitted.

For any $I \in \mathcal{I}_2$, from the proof of Lemma 4.3.5 for (ii) $\Rightarrow$ (iii), we have $(\xi_c + L_3) \vee P_{I^m_{\xi_c,\xi_a,\xi_b}} \leq P_I \leq (\xi_a - a + L_1) \wedge P_{I^M_{\xi_c,\xi_a,\xi_b}}$, where $(\xi_c, \xi_a, \xi_b) = (I(c), I(a), I(b)) \in \Xi_{c,a,b}$. By the definition (4.34) of P_{ξ_c,ξ_a,ξ_b}, it is easy to check $P_I \leq P_{I(c),I(a),I(b)}$ for $0 \leq \lambda < 1/2$ and $P_I \geq P_{I(c),I(a),I(b)}$ for $1/2 < \lambda \leq 1$. Therefore, we have $(2\lambda - 1)P_I \geq (2\lambda - 1)P_{I(c),I(a),I(b)}$, and

$$
\begin{aligned}
V(I) &= \lambda a + (2\lambda - 1)P_I - \lambda I(a) + (1-\lambda)I(b) \\
&\geq \lambda a + (2\lambda - 1)P_{I(c),I(a),I(b)} - \lambda I(a) + (1-\lambda)I(b) = w(I(c), I(a), I(b)).
\end{aligned}
$$

Thus $\min_{I\in\mathcal{I}_2} V(I) \geq \min_{(\xi_c,\xi_a,\xi_b)\in\Xi_{c,a,b}} w(\xi_c,\xi_a,\xi_b)$.

On the contrary, for any $(\xi_c,\xi_a,\xi_b)\in\Xi_{c,a,b}$, using similar arguments to those used in the proof of Lemma 4.2.3, we know that there exists $I\in\mathcal{I}$ such that $P_I = P_{\xi_c,\xi_a,\xi_b}$, $I(c)=\xi_c$, $I(a)=\xi_a$ and $I(b)=\xi_b$. Thus, I satisfies $\xi_c+L_3\leq P_I\leq \xi_a+L_1-a$, namely $I\in\mathcal{I}_2$ and $V(I)=w(\xi_c,\xi_a,\xi_b)$. It implies that $\min_{I\in\mathcal{I}_2} V(I)\leq \min_{(\xi_c,\xi_a,\xi_b)\in\Xi_{c,a,b}} w(\xi_c,\xi_a,\xi_b)$. Thus, $\min_{I\in\mathcal{I}_2} V(I) = \min_{(\xi_c,\xi_a,\xi_b)\in\Xi_{c,a,b}} w(\xi_c,\xi_a,\xi_b)$. ■

Proof of Lemma 4.3.7. **(a)** Assume $a<b$ and $0\leq\lambda<1/2$. For any $(\xi_c,\xi_a)\in\Xi_c\times\Xi_{a,\xi_c}$ where $\Xi_c=[\xi_c^m,\,\xi_c^M]$ and $\Xi_{a,\xi_c}=[\xi_a^m(\xi_c),\,\xi_a^M(\xi_c)]$, in the first step, we solve the problem of $\min_{\xi_b\in\Xi_{b,\xi_c,\xi_a}} w(\xi_c,\xi_a,\xi_b)$, where $\Xi_{b,\xi_c,\xi_a}=[\xi_b^m(\xi_c,\xi_a),\,\xi_b^M(\xi_c,\xi_a)]$. By Lemma 4.3.6, we have

$$\begin{aligned} w(\xi_c,\xi_a,\xi_b) &= \lambda a + (2\lambda-1)\left[(L_1-a+\xi_a)\wedge P_{I^M_{\xi_c,\xi_a,\xi_b}}\right] - \lambda\xi_a + (1-\lambda)\xi_b \\ &= \lambda a - \lambda\xi_a + \lambda\xi_b + (1-2\lambda)\left[(\xi_b - L_1 + a - \xi_a)\vee\left(\xi_b - P_{I^M_{\xi_c,\xi_a,\xi_b}}\right)\right], \end{aligned}$$

thus $w(\xi_c,\xi_a,\xi_b)$ inherits the increment in $\xi_b\in\Xi_{b,\xi_c,\xi_a}$ from the function $\xi_b - P_{I^M_{\xi_c,\xi_a,\xi_b}}$ by Proposition 4.3.2(a). Therefore, the minimizer of $\min_{\xi_b\in\Xi_{b,\xi_c,\xi_a}} w(\xi_c,\xi_a,\xi_b)$, is the left-end point $\xi_b^*(\xi_c,\xi_a)=\xi_b^m(\xi_c,\xi_a)$ of the set Ξ_{b,ξ_c,ξ_a}.

In the second step, we solve the problem of $\displaystyle\min_{\xi_a\in\Xi_{a,\xi_c}} w(\xi_c,\xi_a,\xi_b^m(\xi_c,\xi_a)) = \min_{\xi_a\in\Xi_{a,\xi_c}} w_2(\xi_c,\xi_a)$. In doing so, consider the supremum of the set $\{\xi_a\in\Xi_{a,\xi_c}:\, A_{\xi_c}(\xi_a)<\xi_c+L_3\}$, denoted by

$$\xi_{a,\xi_c} = \sup\left\{\xi_a\in\Xi_{a,\xi_c}:\, A_{\xi_c}(\xi_a)<\xi_c+L_3\right\}. \tag{4.48}$$

By convention, the supremum (4.48) is defined as the left-end point $\xi_a^m(\xi_c)$ of the set Ξ_{a,ξ_c} if the set $\{\xi_a\in\Xi_{a,\xi_c}:\, A_{\xi_c}(\xi_a)<\xi_c+L_3\}$ is empty. Note that $A_{\xi_c}(\xi_a)$ is continuous and strictly increasing in ξ_a, thus there are three possible scenarios for the supremum (4.48). First of all, if $\xi_c+L_3\leq A_{\xi_c}(\xi_a^m(\xi_c))$, then $\xi_{a,\xi_c}=\xi_a^m(\xi_c)$. Secondly, if $A_{\xi_c}(\xi_a^m(\xi_c))<\xi_c+L_3<A_{\xi_c}(\xi_a^M(\xi_c))$, then $\xi_a^m(\xi_c)<\xi_{a,\xi_c}<\xi_a^M(\xi_c)$ and $A_{\xi_c}(\xi_{a,\xi_c})=\xi_c+L_3$. The last scenario is that if $A_{\xi_c}(\xi_a^M(\xi_c))\leq\xi_c+L_3$, then $\xi_{a,\xi_c}=\xi_a^M(\xi_c)$. In the following, we discuss the properties of the function $w_2(\xi_c,\xi_a)$ in the second scenario, that is to assume $A_{\xi_c}(\xi_a^m(\xi_c))<\xi_c+L_3<A_{\xi_c}(\xi_a^M(\xi_c))$.

Case a.1. For $\xi_a^m(\xi_c)\leq\xi_a\leq\xi_{a,\xi_c}$, we have $A_{\xi_c}(\xi_a)\leq\xi_c+L_3$, and then $P_{I^M_{\xi_c,\xi_a,\xi_a}} = A_{\xi_c}(\xi_a)\leq\xi_c+L_3$. By (4.29), we have $P_{I^M_{\xi_c,\xi_a,\xi_a+b-a}}\geq\xi_c+L_3$. Since $P_{I^M_{\xi_c,\xi_a,\xi_b}}$ is continuous and strictly increasing in ξ_b, we know that the equation $P_{I^M_{\xi_c,\xi_a,\xi_b}}=\xi_c+L_3$ has a unique solution $\xi_{b,0}\in[\xi_a,\xi_a+b-a]$, namely, (4.25) is satisfied by $(\xi_c,\xi_a,\xi_{b,0})$. Meanwhile, (4.26) is satisfied by $(\xi_c,\xi_a,\xi_{b,0})$ because $P_{I^m_{\xi_c,\xi_a,\xi_{b,0}}}\leq P_{I^M_{\xi_c,\xi_a,\xi_{b,0}}}=\xi_c+L_3\leq L_1-a+\xi_a$. Thus,

$(\xi_c, \xi_a, \xi_{b,0}) \in \Xi_{c,a,b}$ and $\xi_{b,0} \in \Xi_{b,\xi_c,\xi_a}$. For any $\xi_b < \xi_{b,0}$, because $P_{I^M_{\xi_c,\xi_a,\xi_b}} < \xi_c + L_3$, namely, (4.25) is not satisfied, we have that $(\xi_c, \xi_a, \xi_b) \notin \Xi_{c,a,b}$ and then $\xi_b \notin \Xi_{b,\xi_c,\xi_a}$. Therefore, $\xi_b^m(\xi_c, \xi_a) = \xi_{b,0}$ and $P_{I^M_{\xi_c,\xi_a,\xi_b^m(\xi_c,\xi_a)}} = \xi_c + L_3$. Now, for any ξ_1 and ξ_2 such that $\xi_a^m(\xi_c) \leq \xi_1 < \xi_2 \leq \xi_{a,\xi_c}$, we have that $\xi_b^m(\xi_c, \xi_i)$ satisfies $P_{I^M_{\xi_c,\xi_i,\xi_b^m(\xi_c,\xi_i)}} = \xi_c + L_3$, for $i = 1, 2$. Then, the equation $P_{I^M_{\xi_c,\xi_1,\xi_b^m(\xi_c,\xi_1)}} = \xi_c + L_3 = P_{I^M_{\xi_c,\xi_2,\xi_b^m(\xi_c,\xi_2)}}$ implies that

$$\int_{a+\xi_b^m(\xi_c,\xi_2)-\xi_2}^{a+\xi_b^m(\xi_c,\xi_1)-\xi_1} S_X(x)\mathrm{d}x = \int_{c+\xi_1-\xi_c}^{c+\xi_2-\xi_c} S_X(x)\mathrm{d}x > 0.$$

Since $S_X(x)$ is positive and decreasing in x, we have $a+\xi_b^m(\xi_c, \xi_1)-\xi_1-(a+\xi_b^m(\xi_c, \xi_2)-\xi_2) \geq c+\xi_2-\xi_c-(c+\xi_1-\xi_c)$ and thus $\xi_b^m(\xi_c, \xi_1) \geq \xi_b^m(\xi_c, \xi_2)$. Moreover, $\xi_b^m(\xi_c, \xi_2) \to \xi_b^m(\xi_c, \xi_1)$ as $\xi_2 \to \xi_1$. Therefore, $\xi_b^m(\xi_c, \xi_a)$ is continuous and decreasing in $\xi_a \in [\xi_a^m(\xi_c), \xi_{a,\xi_c}]$. Since $\xi_b^*(\xi_c, \xi_a) = \xi_b^m(\xi_c, \xi_a)$ and $P_{I^M_{\xi_c,\xi_a,\xi_b^*(\xi_c,\xi_a)}} = \xi_c + L_3 \leq L_1 - a + \xi_a$, we have

$$\begin{aligned} w_2(\xi_c, \xi_a) &= \lambda a + (2\lambda - 1) \left[(L_1 - a + \xi_a) \wedge P_{I^M_{\xi_c,\xi_a,\xi_b^*(\xi_c,\xi_a)}} \right] - \lambda\xi_a + (1-\lambda)\xi_b^*(\xi_c, \xi_a) \\ &= \lambda a + (2\lambda - 1)(\xi_c + L_3) - \lambda\xi_a + (1-\lambda)\xi_b^*(\xi_c, \xi_a) \end{aligned}$$

is continuous and decreasing in $\xi_a \in [\xi_a^m(\xi_c), \xi_{a,\xi_c}]$. In particular, when $\xi_a = \xi_{a,\xi_c}$, it is easy to check that the equation $A_{\xi_c}(\xi_{a,\xi_c}) = \xi_c + L_3$ implies that $\xi_b^m(\xi_c, \xi_{a,\xi_c}) = \xi_{a,\xi_c}$, and $w_2(\xi_c, \xi_{a,\xi_c}) = \lambda a + (1 - 2\lambda) (\xi_{a,\xi_c} - (\xi_c + L_3))$.

Case a.2. For $\xi_{a,\xi_c} < \xi_a \leq \xi_a^M(\xi_c)$, we have that $A_{\xi_c}(\xi_a) > \xi_c + L_3$, then (4.25) is satisfied by (ξ_c, ξ_a, ξ_a). Since $\xi_a \in \Xi_{a,\xi_c}$, (4.30) implies that (ξ_c, ξ_a, ξ_a) satisfies (4.26). Thus, $(\xi_c, \xi_a, \xi_a) \in \Xi_{c,a,b}$. It implies that $\xi_a \in \Xi_{b,\xi_c,\xi_a}$ and then $\xi_b^m(\xi_c, \xi_a) = \xi_a$. We have $\xi_b^*(\xi_c, \xi_a) = \xi_a$ and

$$\begin{aligned} w_2(\xi_c, \xi_a) &= \lambda a + (2\lambda - 1) \left[(L_1 - a + \xi_a) \wedge P_{I^M_{\xi_c,\xi_a,\xi_b^*(\xi_c,\xi_a)}} \right] - \lambda\xi_a + (1-\lambda)\xi_b^*(\xi_c, \xi_a) \\ &= \lambda a + (1 - 2\lambda) \left[(a - L_1) \vee (\xi_a - A_{\xi_c}(\xi_a)) \right], \end{aligned}$$

which inherits the continuity and increment in ξ_a form the function $\xi_a - A_{\xi_c}(\xi_a)$ by Proposition 4.3.2(b). Note that $\xi_c + L_3 \leq \xi_{a,\xi_c} - a + L_1$ and then

$$\begin{aligned} \lim_{\xi_a \downarrow \xi_{a,\xi_c}} w_2(\xi_c, \xi_a) &= \lambda a + (1 - 2\lambda) \left[(a - L_1) \vee (\xi_{a,\xi_c} - A_{\xi_c}(\xi_{a,\xi_c})) \right] \\ &= \lambda a + (1 - 2\lambda) \left[(a - L_1) \vee (\xi_{a,\xi_c} - (\xi_c + L_3)) \right] \\ &= \lambda a + (1 - 2\lambda) (\xi_{a,\xi_c} - (\xi_c + L_3)) = w_2(\xi_c, \xi_{a,\xi_c}). \end{aligned}$$

By combining **Cases a.1** and **Case a.2**, we obtain that, when $A_{\xi_c}(\xi_a^m(\xi_c)) < \xi_c + L_3 < A_{\xi_c}(\xi_a^M(\xi_c))$, the function $w_2(\xi_c, \xi_a)$ is continuous in $\xi_a \in \Xi_{a,\xi_c}$ and minimized at the point $\xi_a^*(\xi_c) = \xi_{a,\xi_c}$. If $\xi_c + L_3 \le A_{\xi_c}(\xi_a^m(\xi_c))$, by using the same arguments in **Case a.2**, we have that $w_2(\xi_c, \xi_a)$ is continuous in $\xi_a \in \Xi_{a,\xi_c}$ and minimized at the point $\xi_a^*(\xi_c) = \xi_{a,\xi_c} = \xi_a^m(\xi_c)$. If $A_{\xi_c}(\xi_a^M(\xi_c)) \le \xi_c + L_3$, by using the same arguments in **Case a.1**, we have that $w_2(\xi_c, \xi_a)$ is continuous in $\xi_a \in \Xi_{a,\xi_c}$ and minimized at the point $\xi_a^*(\xi_c) = \xi_{a,\xi_c} = \xi_a^M(\xi_c)$. In short, we conclude that $w_2(\xi_c, \xi_a)$ is continuous in $\xi_a \in \Xi_{a,\xi_c}$ and minimized at the point $\xi_a^*(\xi_c) = \xi_{a,\xi_c}$.

In the last step, we solve the problem of $\min_{\xi_c \in \Xi_c} w_2(\xi_c, \xi_a^*(\xi_c)) = \min_{\xi_c \in \Xi_c} w_1(\xi_c)$. Note that, for each $\xi_c \in \Xi_c$, $\xi_a^M(\xi_c) = \xi_c + a - c$ and $\xi_a^m(\xi_c) \ge \xi_c + L_3 + a - L_1$. By Proposition 4.3.2(b), we know that $\xi_a - A_{\xi_c}(\xi_a)$ and $\xi_a - A^m_{\xi_c}(\xi_a)$ are both continuous and strictly increasing in ξ_a. Consider the following two cases.

Case a.i. If $L_3 \le h_5(\xi_c)$, namely $\xi_c + L_3 \le A_{\xi_c}(\xi_c + L_3 + a - L_1)$, note that $\xi_a^m(\xi_c) \ge \xi_c + L_3 + a - L_1$ and $A_{\xi_c}(\xi_a)$ is increasing on Ξ_{a,ξ_c}, thus $A_{\xi_c}(\xi_a^m(\xi_c)) \ge A_{\xi_c}(\xi_c + L_3 + a - L_1) \ge \xi_c + L_3$. It implies that the set $\{\xi_a \in \Xi_{a,\xi_c} : A_{\xi_c}(\xi_a) < \xi_c + L_3\}$ is empty. Thus, we have $\xi_a^*(\xi_c) = \xi_a^m(\xi_c)$ and $A_{\xi_c}(\xi_a^*(\xi_c)) \ge \xi_c + L_3$. From the arguments in **Case a.2**, we have $\xi_b^*(\xi_c, \xi_a^*(\xi_c)) = \xi_a^*(\xi_c)$ and

$$w_2(\xi_c, \xi_a) = \lambda a + (1 - 2\lambda)\left[(a - L_1) \vee (\xi_a^*(\xi_c) - A_{\xi_c}(\xi_a^*(\xi_c)))\right].$$

Suppose $A_{\xi_c}(\xi_a^*(\xi_c)) < \xi_a^*(\xi_c) + L_1 - a$, then $\xi_a^*(\xi_c) > A_{\xi_c}(\xi_a^*(\xi_c)) - L_1 + a \ge \xi_c + L_3 + a - L_1$ and $\xi_a^*(\xi_c) - A^m_{\xi_c}(\xi_a^*(\xi_c)) \ge \xi_a^*(\xi_c) - A_{\xi_c}(\xi_a^*(\xi_c)) > a - L_1$. Note that $\xi_a - A^m_{\xi_c}(\xi_a)$ is continuous and increasing in $\xi_a \in \Xi_{a,\xi_c}$, then there exists $\xi \in [\xi_c + L_3 + a - L_1,\ \xi_a^*(\xi_c))$ such that $\xi - A^m_{\xi_c}(\xi) > a - L_1$, which implies that ξ satisfies (4.29). Moreover, ξ satisfies (4.30) because $\xi_c + L_3 \le A_{\xi_c}(\xi_c + L_3 + a - L_1) \le A_{\xi_c}(\xi) \le A^M_{\xi_c}(\xi)$. Conditions (4.29) and (4.30) imply $\xi \in \Xi_{a,\xi_c}$, namely $\xi \ge \xi_a^m(\xi_c)$, which contradicts the fact that $\xi < \xi_a^*(\xi_c) = \xi_a^m(\xi_c)$. Therefore, $A_{\xi_c}(\xi_a^*(\xi_c)) \ge \xi_a^*(\xi_c) + L_1 - a$ and $w_2(\xi_c, \xi_a) = \lambda a + (1 - 2\lambda)(a - L_1)$ is a constant function.

Case a.ii. If $L_3 > h_5(\xi_c)$, namely $\xi_c + L_3 > A_{\xi_c}(\xi_c + a - L_1 + L_3)$. Since $A_{\xi_c}(\xi_c + b - c) \ge \xi_c + L_3$, by (4.27), and the fact that $A_{\xi_c}(\xi_a)$ is continuous and strictly increasing in $\xi_a \in \Xi_{a,\xi_c}$, we see that there exists $\xi_{a,1} \in [\xi_c + a - L_1 + L_3,\ \xi_c + b - c]$, which is the unique solution to the equation of $A_{\xi_c}(\xi_{a,1}) = L_3 + \xi_c$. Thus, $\xi_a^*(\xi_c) = \xi_{a,1} \wedge \xi_a^M(\xi_c) = \xi_{a,1} \wedge (\xi_c + a - c) \le \xi_{a,1}$ and $A_{\xi_c}(\xi_a^*(\xi_c)) \le A_{\xi_c}(\xi_{a,1}) = \xi_c + L_3$. Consider the contract $I(x) = x - (x - \xi_c)^+ + (x - c)^+ - (x - (c + \xi_{a,1} - \xi_c))^+ + (x - b)^+$, it is easy to check that $I(c) = \xi_c$, $I(a) = (\xi_c + a - c) \wedge \xi_{a,1} = \xi_a^*(\xi_c)$, $I(b) = \xi_{a,1}$, and $P_I = A_{\xi_c}(\xi_{a,1}) = \xi_c + L_3$. Since $I(c) + L_3 = \xi_c + L_3 = P_I = (\xi_c + L_3 + a - L_1) - a + L_1 \le I(a) - a + L_1$, we have $I \in \mathcal{I}_2$ and $(\xi_c, (\xi_c + a - c) \wedge \xi_{a,1}, \xi_{a,1}) \in \Xi_{c,a,b}$. For any $\xi_b < \xi_{a,1}$, we have $(\xi_c, (\xi_c + a - c) \wedge \xi_{a,1}, \xi_b) \notin \Xi_{c,a,b}$ because either (4.24) is invalid when $\xi_{a,1} < \xi_c + a - c$, or (4.25) is invalid when $\xi_{a,1} \ge \xi_c + a - c$

from the observation that $P_{I^M_{\xi_c,\xi_c+a-c,\xi_b}} = A_{\xi_c}(\xi_b) < A_{\xi_c}(\xi_{a,1}) = \xi_c + L_3$. It implies that $\xi_b \notin \Xi_{b,\xi_c,\xi_a^*(\xi_c)}$ for any $\xi_b < \xi_{a,1}$ and then $\xi_b^*(\xi_c,\xi_a^*(\xi_c)) = \xi_b^m(\xi_c,\xi_a^*(\xi_c)) = \xi_{a,1}$. It is easy to check that, $P_{I^M_{\xi_c,(\xi_c+a-c)\wedge\xi_{a,1},\xi_{a,1}}} = A_{\xi_c}(\xi_{a,1}) = \xi_c + L_3 \le \xi_{a,1} - a + L_1$. Thus,

$$\begin{aligned}
w_1(\xi_c) &= \lambda a - \lambda\xi_a^*(\xi_c) + (1-\lambda)\xi_b^*(\xi_c,\xi_a^*(\xi_c)) + (2\lambda-1)\left[(L_1 - a + \xi_a^*(\xi_c)) \wedge P_{I^M_{\xi_c,\xi_a^*(\xi_c),\xi_b^*(\xi_c,\xi_a^*(\xi_c))}}\right] \\
&= \lambda a - \lambda((\xi_c + a - c)\wedge\xi_{a,1}) + (1-\lambda)\xi_{a,1} + (2\lambda-1)(\xi_c + L_3)
\end{aligned}$$

and it has derivative on the set $\Xi_c \setminus \{\xi_c :\, A_{\xi_c}(\xi_c + a - c) = L_3 + \xi_c + a - c\}$ with

$$\begin{aligned}
w_1'(\xi_c) &= [1 - \lambda - \lambda\,\mathbb{I}(\xi_{a,1} < \xi_c + a - c)]\left(\frac{\mathrm{d}}{\mathrm{d}\xi_c}\xi_{a,1} - 1\right) \\
&= \frac{1}{S_X(c + \xi_{a,1} - \xi_c)}\left(\frac{1}{1+\theta} - S_X(\xi_c)\right)[1 - \lambda - \lambda\,\mathbb{I}(\xi_{a,1} < \xi_c + a - c)],
\end{aligned}$$

where $\frac{\mathrm{d}}{\mathrm{d}\xi_c}\xi_{a,1} = 1 + \left[\frac{1}{1+\theta} - S_X(\xi_c)\right]/S_X(c+\xi_{a,1}-\xi_c)$ since $\xi_{a,1}$ satisfies the equation $\xi_c + L_3 = A_{\xi_c}(\xi_{a,1})$. Note that $w_1'(\xi_c) \le 0 \iff \frac{1}{1+\theta} \le S_X(\xi_c) \iff \xi_c \le v_\theta$.

By combining **Case a.i** and **Case a.ii**, we obtain that $w_1'(\xi_c) \le 0$ when $\xi_c \le v_\theta$ and $w_1'(\xi_c) \ge 0$ when $\xi_c > v_\theta$. Therefore, $\xi_c^* = \xi_c^m \vee \left(v_\theta \wedge \xi_c^M\right)$.

(b) Assume $a < b$ and $1/2 < \lambda \le 1$. By Lemma 4.3.6, we have

$$w(\xi_c,\xi_a,\xi_b) = \lambda a + (2\lambda-1)\left[(\xi_c + L_3) \vee P_{I^m_{\xi_c,\xi_a,\xi_b}}\right] - \lambda\xi_a + (1-\lambda)\xi_b,$$

where $P_{I^m_{\xi_c,\xi_a,\xi_b}}$ is increasing in ξ_b. Thus, $w(\xi_c,\xi_a,\xi_b)$ is continuous and increasing in ξ_b. It implies that $\xi_b^*(\xi_c,\xi_a) = \xi_b^m(\xi_c,\xi_a)$ and

$$w_2(\xi_c,\xi_a) = \min_{\xi_b\in\Xi_{b,\xi_c,\xi_a}} w(\xi_c,\xi_a,\xi_b) = w(\xi_c,\xi_a,\xi_b^m(\xi_c,\xi_a)),$$

where $\Xi_{b,\xi_c,\xi_a} = [\xi_b^m(\xi_c,\xi_a), \xi_b^M(\xi_c,\xi_a)]$. As discussed in **(a)**, there are three possible scenarios for ξ_{a,ξ_c} defined by (4.48), and we consider the following two cases under the assumption that $A_{\xi_c}(\xi_a^m(\xi_c)) < \xi_c + L_3 < A_{\xi_c}(\xi_a^M(\xi_c))$.

Case b.1. If $\xi_a^m(\xi_c) \le \xi_a \le \xi_{a,\xi_c}$, then $A_{\xi_c}(\xi_a) \le \xi_c + L_3$ and from the arguments in **Case a.1**, we know that $\xi_b^*(\xi_c,\xi_a)$ is the solution to the equation of $\xi_c + L_3 = P_{I^M_{\xi_c,\xi_a,\xi_b^*(\xi_c,\xi_a)}}$ and is decreasing in ξ_a. It follows that

$$\begin{aligned}
w_2(\xi_c,\xi_a) &= \lambda a + (2\lambda-1)\left[(\xi_c + L_3) \vee P_{I^m_{\xi_c,\xi_a,\xi_b^*(\xi_c,\xi_a)}}\right] - \lambda\xi_a + (1-\lambda)\xi_b^*(\xi_c,\xi_a) \\
&= \lambda a + (2\lambda-1)(\xi_c + L_3) - \lambda\xi_a + (1-\lambda)\xi_b^*(\xi_c,\xi_a)
\end{aligned}$$

is continuous and decreasing in ξ_a. In particular, when $\xi_a = \xi_{a,\xi_c}$, we have that $w_2(\xi_c, \xi_{a,\xi_c}) = \lambda a + (2\lambda - 1)(\xi_c + L_3 - \xi_{a,\xi_c})$ where $\xi_b^*(\xi_c, \xi_{a,\xi_c}) = \xi_{a,\xi_c}$ is induced from the equation $A_{\xi_c}(\xi_{a,\xi_c}) = \xi_c + L_3$.

Case b.2. If $\xi_{a,\xi_c} < \xi_a \leq \xi_a^M(\xi_c)$, then $A_{\xi_c}(\xi_a) > \xi_c + L_3$ and from the arguments used in **Case a.2**, we have $\xi_b^*(\xi_c, \xi_a) = \xi_a$. Since $\xi_a - A_{\xi_c}^m(\xi_a)$ is increasing in ξ_a, we see that

$$\begin{aligned} w_2(\xi_c, \xi_a) &= \lambda a + (2\lambda - 1)\left[(\xi_c + L_3) \vee P_{I^m_{\xi_c,\xi_a,\xi_a}}\right] + (1 - 2\lambda)\xi_a \\ &= \lambda a + (2\lambda - 1)\left[(\xi_c + L_3 - \xi_a) \vee \left(A_{\xi_c}^m(\xi_a) - \xi_a\right)\right] \end{aligned}$$

is continuous and decreasing in ξ_a. Moreover, $A_{\xi_c}^m(\xi_{a,\xi_c}) < A_{\xi_c}(\xi_{a,\xi_c}) = \xi_c + L_3$ implies that $\lim_{\xi_a \downarrow \xi_{a,\xi_c}} w_2(\xi_c, \xi_a) = w_2(\xi_c, \xi_{a,\xi_c})$.

By combining **Case b.1** and **Case b.2**, we see that $w_2(\xi_c, \xi_a)$ is decreasing in ξ_a and $\xi_a^*(\xi_c) = \xi_a^M(\xi_c) = \xi_c + a - c$. In the other two scenarios, by using the same argument as in **Case b.1** and **Case b.2**, we can show that $w_2(\xi_c, \xi_a)$ is decreasing in ξ_a and $\xi_a^*(\xi_c) = \xi_a^M(\xi_c) = \xi_c + a - c$.

Next, we will solve the minimization problem of $\min_{\xi_c \in \Xi_c} w_1(\xi_c) = \min_{\xi_c \in \Xi_c} w_2(\xi_c, \xi_a^*(\xi_c)) = \min_{\xi_c \in \Xi_c} w_2(\xi_c, \xi_c + a - c)$. Note that $h_3(\xi_c) = A_{\xi_c}(\xi_c + a - c) - \xi_c = A_{\xi_c}(\xi_a^*(\xi_c)) - \xi_c$.

If $h_3(\xi_c) < L_3$, namely $A_{\xi_c}(\xi_a^*(\xi_c)) < \xi_c + L_3$, then, from **Case b.1**, the value function $w_1(\xi_c)$ is reduced to $w_1(\xi_c) = \lambda a + (2\lambda - 1)(\xi_c + L_3) - \lambda(\xi_c + a - c) + (1 - \lambda)\xi_b^*(\xi_c, \xi_c + a - c)$, where $\xi_b^*(\xi_c, \xi_a^*(\xi_c))$ is the solution to the equation

$$\xi_c + L_3 = P_{I^M_{\xi_c, \xi_a^*(\xi_c), \xi_b^*(\xi_c, \xi_a^*(\xi_c))}} = (1 + \theta)\left(\int_0^{\xi_c} + \int_c^{c + \xi_b^*(\xi_c, \xi_a^*(\xi_c)) - \xi_c} + \int_b^{\infty}\right) S_X(x)\mathrm{d}x.$$

Thus, $\xi_b^*(\xi_c, \xi_a^*(\xi_c))$ is continuous in ξ_c and $\frac{\mathrm{d}}{\mathrm{d}\xi_c}\xi_b^*(\xi_c, \xi_a^*(\xi_c)) = 1 + \frac{\frac{1}{1+\theta} - S_X(\xi_c)}{S_X(c + \xi_b^*(\xi_c, \xi_a^*(\xi_c)) - \xi_c)}$.

If $h_3(\xi_c) \geq L_3$, namely $A_{\xi_c}(\xi_a^*(\xi_c)) \geq \xi_c + L_3$, then, from **Case b.2**, we have $\xi_b^*(\xi_c, \xi_a^*(\xi_c)) = \xi_a^*(\xi_c) = \xi_c + a - c$ and

$$\begin{aligned} w_1(\xi_c) &= \lambda a + (2\lambda - 1)\left\{[\xi_c + L_3 - \xi_a^*(\xi_c)] \vee \left[A_{\xi_c}^m(\xi_a^*(\xi_c)) - \xi_a^*(\xi_c)\right]\right\} \\ &= \lambda a + (1 - 2\lambda)(a - c) + (2\lambda - 1)\left[L_3 \vee h_2(\xi_c)\right], \end{aligned}$$

where $h_2(\xi_c) = A_{\xi_c}(\xi_c + a - c) - \xi_c$.

Therefore, $w_1(\xi_c)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\xi_c} w_1(\xi_c) = \begin{cases} (2\lambda - 1)\left((1 + \theta)S_X(c - \xi_c) - 1\right), & \text{if } L_3 < h_2(\xi_c), \\ 0, & \text{if } h_2(\xi_c) < L_3 < h_3(\xi_c), \\ \frac{1 - \lambda}{S_X(c + \xi_b^*(\xi_c, \xi_a^*(\xi_c)) - \xi_c)}\left(\frac{1}{1+\theta} - S_X(\xi_c)\right), & \text{if } h_3(\xi_c) < L_3. \end{cases} \tag{4.49}$$

By Proposition 4.3.1(b) and Proposition 4.3.2(c), we have that h_2 is convex and achieves it minimal value at $c - v_\theta$ while h_3 is concave and achieves its maximal value at v_θ. Thus, $\xi_{L_3,h_2} = \sup\{\xi_c \in [0, c - v_\theta] : L_3 \le h_2(\xi_c)\}$ and $\xi_{L_3,h_3} = \sup\{\xi_c \in [0, v_\theta] : h_3(\xi_c) \le L_3\}$ are both well-defined. Now consider the following three cases.

Case b.i. Suppose $L_3 \le h_2(0) < h_3(0)$. Firstly, since h_2 is decreasing on $[0, c - v_\theta]$ and h_3 is increasing on $[0, v_\theta]$, we have $\xi_{L_3,h_3} = 0 \le \xi_{L_3,h_2}$. Secondly, since $L_3 \le L_1 - c$ and h_2 is decreasing on $[0, c - v_\theta]$, we have $\sup\{\xi_c \in [0, c - v_\theta] : L_1 - c \le h_2(\xi_c)\} \le \xi_{L_3,h_2} \le c - v_\theta$, where $h_2(c - v_\theta) \le L_1 - c$ due to (4.7). Thus, $h_2(\xi_{L_3,h_2}) \le L_1 - c$, and moreover, $L_3 \le h_2(\xi_{L_3,h_2}) \le h_1(\xi_{L_3,h_2})$. Thus, ξ_{L_3,h_2} satisfies (4.27) and (4.28), namely $\xi_{L_3,h_2} \in \Xi_c$. For any $\xi_c \le \xi_{L_3,h_2}$, we have $h_2(\xi_c) \ge h_2(\xi_{L_3,h_2}) \ge L_3$, and thus, from (4.49), $w'(\xi_c) = (2\lambda - 1)\left[(1+\theta)S_X(c - \xi_c) - 1\right] \le 0$. If $\xi_{L_3,h_2} < \xi_c$ and $L_3 < h_2(\xi_c)$, then $\xi_c \ge c - v_\theta$ from the observation that $h_2(\xi) \le h_2(\xi_{L_3,h_2}) = L_3$ for any $\xi_{L_3,h_2} < \xi < c - v_\theta$. Then $w'(\xi_c) = (2\lambda - 1)\left[(1+\theta)S_X(c - \xi_c) - 1\right] \ge 0$. If $\xi_{L_3,h_2} < \xi_c$ and $h_2(\xi_c) < L_3 < h_3(\xi_c)$, then $w_2'(\xi_c) = 0$. If $\xi_{L_3,h_2} < \xi_c$ and $h_3(\xi_c) < L_3$, then $\xi_c \ge v_\theta$ from the observation that $h_3(\xi) \ge h_3(0) > 0$ for any $0 < \xi \le v_\theta$, then $w_2'(\xi_c) = \frac{1-\lambda}{S_X(c + \xi_b^*(\xi_c, \xi_a^*(\xi_c)) - \xi_c)}\left(\frac{1}{1+\theta} - S_X(\xi_c)\right) \ge 0$. In short, $w_2(\xi_c) \le 0$ for any $\xi_c \le \xi_{L_3,h_2}$, and $w_2(\xi_c) \ge 0$ for any $\xi_c > \xi_{L_3,h_2}$. Thus, we conclude that $w_1(\xi_c)$ achieves its minimal value at the point $\xi_c^* = \xi_{L_3,h_2}$.

Case b.ii. Suppose $L_3 \ge h_3(0) > h_2(0)$. Using similar arguments as in **Case b.i**, we conclude that $\xi_{L_3,h_2} = 0 \le \xi_{L_3,h_3}$, $\xi_{L_3,h_3} \in \Xi_c$, and $w_1(\xi_c)$ achieves its minimal value at $\xi_c^* = \xi_{L_3,h_3}$.

Case b.iii. Suppose $h_2(0) < L_3 < h_3(0)$, then $\xi_{L_3,h_2} = \xi_{L_3,h_3} = 0$. Inequalities $h_1(0) \ge h_3(0) > L_3$ and $h_2(0) < L_3 \le L_1 - c$ imply that 0 satisfies (4.27) and (4.28) and thus $0 \in \Xi_c$. Thus, it follows from **Case b.i** and **Case b.ii** that $w'(\xi_c) \ge 0$ for all $\xi_c \in [0, c]$ and thus $\xi_c^* = 0$.

By combining **Case b.i**, **Case b.ii**, and **Case b.iii**, we obtain $\xi_c^* = \xi_{L_3,h_2} \vee \xi_{L_3,h_3}$.

(c) Assume $b < a$ and $0 \le \lambda < 1/2$. By Lemma 4.3.6, we have that

$$\begin{aligned} w(\xi_c, \xi_a, \xi_b) =&\ \lambda a + (2\lambda - 1)\left[(L_1 - a + \xi_a) \wedge P_{I^M_{\xi_c,\xi_a,\xi_b}}\right] - \lambda \xi_a + (1-\lambda)\xi_b \\ =&\ \lambda a + (1-\lambda)\xi_b - (1-\lambda)\left[(L_1 - a + \xi_a) \wedge P_{I^M_{\xi_c,\xi_a,\xi_b}}\right] - \lambda\left[(a - L_1) \vee \left(\xi_a - P_{I^M_{\xi_c,\xi_a,\xi_b}}\right)\right] \end{aligned}$$

is continuous in $\xi_a \in \Xi_{a,\xi_c,\xi_b}$. Note that $\xi_a - P_{I^M_{\xi_c,\xi_a,\xi_b}}$ is increasing in ξ_a by Proposition 4.3.3(a). Thus, $w(\xi_c, \xi_a, \xi_b)$ is decreasing in ξ_a and $\xi_a^*(\xi_c, \xi_b) = \xi_a^M(\xi_c, \xi_b) = \xi_b + a - b$. Next, we will solve the problem of $\min_{\xi_b\in\Xi_{b,\xi_c}} w_2(\xi_c,\xi_b)$, where

$$\begin{aligned}
w_2(\xi_c,\xi_b) &= \min_{\xi_a\in\Xi_{a,\xi_c,\xi_b}} w(\xi_c,\xi_a,\xi_b) = w(\xi_c,\xi_a^*(\xi_c,\xi_b),\xi_b)\\
&= \lambda a - \lambda(\xi_b + a - b) + (1-\lambda)\xi_b + (2\lambda-1)\left[(L_1 - b + \xi_b)\wedge P_{I^M_{\xi_c,\xi_b+a-b,\xi_b}}\right]\\
&= \lambda b + (1-2\lambda)\left[(b-L_1)\vee\left(\xi_b - B^M_{\xi_c}(\xi_b)\right)\right].
\end{aligned}$$

The function $\xi_b - B^M_{\xi_c}(\xi_b)$ is increasing in ξ_b by Proposition 4.3.3(b) and thus $\xi_b^*(\xi_c) = \xi_b^m(\xi_c)$. Note that (4.24) implies $\xi_b^*(\xi_c) \geq \xi_c + (L_3 - L_1 + b)^+$. Moreover, from (4.31) and (4.32), $\xi_b \in \Xi_{b,\xi_c}$ is equivalent to $\xi_c + L_3 \leq B^M_{\xi_c}(\xi_b)$ and $b - L_1 \leq \xi_b - B^m_{\xi_c}(\xi_b)$. Consider the following three cases.

Case c.1. Suppose $h_7(\xi_c) \geq L_3$ and $h_6(\xi_c) \leq L_3 \vee (L_1 - b)$, which mean $\xi_c + L_3 \leq B^M_{\xi_c}(\xi_c + (b - L_1 + L_3)^+)$ and $B^m_{\xi_c}(\xi_c + (b-L_1+L_3)^+) \leq \xi_c + (b - L_1 + L_3)^+ + L_1 - b$. Thus, $\xi_b^*(\xi_c) = \xi_c + (b + L_3 - L_1)^+ \in \Xi_{b,\xi_c}$ and

$$\begin{aligned}
\min_{\xi_c\in\Xi_c} w_1(\xi_c) &= \min_{\xi_c\in\Xi_c} w_2(\xi_c,\xi_b^*(\xi_c)) = \min_{\xi_c\in\Xi_c}\left\{\lambda b + (1-2\lambda)\left[(b-L_1)\vee\left((b-L_3-L_1)^+ - h_7(\xi_c)\right)\right]\right\}\\
&= \lambda b + (1-2\lambda)\left[(b-L_1)\vee\left((b+L_3-L_1)^+ - \max_{\xi_c\in\Xi_c} h_7(\xi_c)\right)\right].
\end{aligned}$$

By Proposition 4.3.3(d), we know that h_7 is increasing on $[0,v_\theta)$ and decreasing on $(v_\theta,c]$, which, together with $0 \leq \lambda < 1/2$, imply that $w_1(\xi_c)$ is decreasing if $\xi_c \in [0,v_\theta)$ and increasing if $\xi_c \in (v_\theta, c]$.

Case c.2. Suppose $h_7(\xi_c) < L_3$, then $h_6(\xi_c) \leq h_7(\xi_c) < L_3 \leq L_3 \vee (L_1 - b)$. It follows that $\xi_c + L_3 > B^M_{\xi_c}(\xi_c + (b-L_1+L_3)^+)$ and $B^m_{\xi_c}(\xi_c+(b-L_1+L_3)^+) \leq \xi_c + (b-L_1+L_3)^+ + L_1 - b$. Note that, $\xi_c \in \Xi_c$ and (4.27) implies $\xi_c + L_3 \leq h_1(\xi_c) = B^M_{\xi_c}(\xi_c + b - c)$. Since $B^M_{\xi_c}(\xi_b)$ is continuous and strictly increasing in ξ_b, there exists $\xi_{b,1} \in [\xi_c + (b - L_1 + L_3)^+, \xi_c + b - c]$ such that $\xi_{b,1}$ is the unique solution to the equation of $\xi_c + L_3 = B^M_{\xi_c}(\xi_{b,1})$, and thus $\xi_{b,1}$ satisfies (4.31). Meanwhile, since $\xi_b - B^m_{\xi_c}(\xi_b)$ is strictly increasing in ξ_b, we have $\xi_{b,1} - B^m_{\xi_c}(\xi_{b,1}) \geq \xi_c + (b - L_1 + L_3)^+ - B^m_{\xi_c}(\xi_c + (b - L_1 + L_3)^+) \geq b - L_1$ which implies that $\xi_{b,1}$ satisfies (4.32). Thus, $\xi_{b,1} \in \Xi_{b,\xi_c}$. For any $\xi_b < \xi_{b,1}$, we have $\xi_b \notin \Xi_{c,\xi_b}$ because $B^M_{\xi_c}(\xi_b) < B^M_{\xi_c}(\xi_{b,1}) = \xi_c + L_3$. Therefore, $\xi_b^*(\xi_c) = \xi_b^m(\xi_c) = \xi_{b,1}$. Moreover,

$$\begin{aligned}
w_1(\xi_c) &= \lambda a + (2\lambda - 1)(\xi_c + L_3) - \lambda\left(\xi_b^*(\xi_c) + a - b\right) + (1-\lambda)\xi_b^*(\xi_c)\\
&= \lambda b + (1-2\lambda)\left(\xi_b^*(\xi_c) - \xi_c - L_3\right),
\end{aligned}$$

which implies that

$$w_1'(\xi_c) = (1-2\lambda)\left(\frac{\mathrm{d}\xi_b^*(\xi_c)}{\mathrm{d}\xi_c} - 1\right) = \frac{1-2\lambda}{S_X(c + \xi_b^*(\xi_c) + \xi_c)}\left(\frac{1}{1+\theta} - S_X(\xi_c)\right),$$

where $\frac{\mathrm{d}}{\mathrm{d}\xi_c}\xi_b^*(\xi_c) = 1 + \left(\frac{1}{1+\theta} - S_X(\xi_c)\right)/S_X(c + \xi_b^*(\xi_c) + \xi_c)$ is induced from the equation $B_{\xi_c}^M(\xi_b^*(\xi_c)) = B_{\xi_c}^M(\xi_{b,1}) = \xi_c + L_3$. Thus, the function $w_1(\xi_c)$ is decreasing if $\xi_c \in [0, v_\theta)$ and increasing if $\xi_c \in (v_\theta, c]$.

Case c.3. Suppose $h_7(\xi_c) \geq L_3$ and $h_6(\xi_c) > L_3 \vee (L_1 - b)$, which imply $\xi_c + L_3 \leq B_{\xi_c}^M(\xi_c + (b - L_1 + L_3)^+)$ and $B_{\xi_c}^m(\xi_c + (b - L_1 + L_3)^+) > \xi_c + (b - L_1 + L_3)^+ + L_1 - b$. By the similar arguments in **Case c.2**, we know that there exists $\xi_{b,0} \in [\xi_c + (b - L_1 + L_3)^+,\ \xi_c + b - c]$ such that $\xi_{b,0}$ is the unique solution to the equation of $L_1 - b + \xi_{b,0} = B_{\xi_c}^m(\xi_{b,0})$ and $\xi_b^*(\xi_c) = \xi_{b,0}$. Hence, $w_1(\xi_c) = \lambda b + (2\lambda - 1)(L_1 - b + \xi_b^*(\xi_c)) + (1 - 2\lambda)\xi_b^*(\xi_c) = (2\lambda - 1)(L_1 - b) + \lambda b$ is a constant function.

By combining **Case c.1**, **Case c.2**, and **Case c.3**, we obtain that $w_1(\xi_c)$ is decreasing if $\xi_c \in [0, v_\theta)$ and increasing if $\xi_c \in (v_\theta, c]$. Therefore, $\xi_c^* = \xi_c^m \vee \left(v_\theta \wedge \xi_c^M\right)$.

(d) Assume $b < a$ and $1/2 < \lambda \leq 1$. By Lemma 4.3.6, we have

$$\begin{aligned} w(\xi_c, \xi_b, \xi_a) &= \lambda a - \lambda \xi_a + (1 - \lambda)\xi_b + (2\lambda - 1)\left[(\xi_c + L_3) \vee P_{I^m_{\xi_c,\xi_a,\xi_b}}\right] \\ &= \lambda a + (1 - \lambda)\xi_b - (1 - \lambda)\xi_a + (2\lambda - 1)\left[(\xi_c + L_3 - \xi_a) \vee \left(P_{I^m_{\xi_c,\xi_a,\xi_b}} - \xi_a\right)\right]. \end{aligned}$$

Thus $w(\xi_c, \xi_a, \xi_b)$ is continuous and decreasing in ξ_a due to the properties of $P_{I^m_{\xi_c,\xi_a,\xi_b}} - \xi_a$ given in Proposition 4.3.3(a). Hence, we have $\xi_a^*(\xi_c, \xi_b) = \xi_a^M(\xi_c, \xi_b) = \xi_b + a - b$ and

$$\begin{aligned} w_2(\xi_c, \xi_b) = w(\xi_c, \xi_a^*(\xi_c, \xi_b), \xi_b) &= \lambda a - \lambda(\xi_b + a - b) + (1 - \lambda)\xi_b + (2\lambda - 1)\left[(\xi_c + L_3) \vee B_{\xi_c}^m(\xi_b)\right] \\ &= \lambda b + (2\lambda - 1)\left[(\xi_c + L_3 - \xi_b) \vee \left(B_{\xi_c}^m(\xi_b) - \xi_b\right)\right]. \end{aligned}$$

Thus $w_2(\xi_c, \xi_b)$ is continuous and decreasing in ξ_b due to the properties of $B_{\xi_c}^m(\xi_b) - \xi_b$ given in Proposition 4.3.3(b). It implies $\xi_b^*(\xi_c) = \xi_b^M(\xi_c) = \xi_c + b - c$ and thus

$$\begin{aligned} \min_{\xi_c \in \Xi_c} w_1(\xi_c) &= \min_{\xi_c \in \Xi_c} w_2(\xi_c, \xi_b^*(\xi_c)) \\ &= \min_{\xi_c \in \Xi_c} \left\{\lambda b + (2\lambda - 1)\left[(c + L_3 - b) \vee \left(B_{\xi_c}^m(\xi_c + b - c) - (\xi_c + b - c)\right)\right]\right\} \\ &= \min_{\xi_c \in \Xi_c} \left\{(1 - \lambda)b + (2\lambda - 1)c + (2\lambda - 1)\left[L_3 \vee h_2(\xi_c)\right]\right\} \\ &= (1 - \lambda)b + (2\lambda - 1)c + (2\lambda - 1)\left[L_3 \vee \min_{\xi_c \in \Xi_c} h_2(\xi_c)\right]. \end{aligned}$$

Since $h_2(\xi_c)$ is continuous, decreasing on $[0, c - v_\theta)$, and increasing on $(c - v_\theta, c]$, we obtain that $w_1(\xi_c)$ is continuous and $\xi_c^* = \xi_c^m \vee \left[(c - v_\theta) \wedge \xi_c^M\right]$. ■

Proof of Theorem 4.3.8. We assume $a < b$. The proof for the case of $a > b$ is similar to the case of $a < b$ and is omitted.

For $a < b$, we have $\xi_b^* = \xi_b^*(\xi_c^*, \xi_a^*) \in \Xi_{b,\xi_c^*,\xi_a^*}$. Note that Ξ_{b,ξ_c^*,ξ_a^*} is the set of all $\xi_b \in [\xi_a^*, \xi_a^* + b - a]$ such that $(\xi_c^*, \xi_a^*, \xi_b) \in \Xi_{c,a,b}$, thus $(\xi_c^*, \xi_a^*, \xi_b^*) \in \Xi_{c,a,b}$. It is easy to check that any contract I of the form

$$\begin{aligned} I(x) = &(x-d_1)^+ - (x - d_1 - \xi_c^*)^+ + (x-d_2)^+ - (x - (d_2 + \xi_a^* - \xi_c^*))^+ \\ &+ (x-d_3)^+ - (x - (d_3 + \xi_b^* - \xi_a^*))^+ + (x-d_4)^+ \end{aligned} \tag{4.50}$$

for some $(d_1, d_2, d_3, d_4) \in [0, c - \xi_c^*] \times [c, a - \xi_a^* + \xi_c^*] \times [a, b - \xi_b^* + \xi_a^*] \times [b, \infty]$, satisfies $I \in \mathcal{I}$, $I(c) = \xi_c^*$, $I(a) = \xi_a^*$, $I(b) = \xi_b^*$ and $I^m_{\xi_c^*,\xi_a^*,\xi_b^*}(x) \le I(x) \le I^M_{\xi_c^*,\xi_a^*,\xi_b^*}(x)$ for all $x \ge 0$. For I of the form (4.50), its premium is given by

$$P_I = P(d_1, d_2, d_3, d_4) = (1+\theta)\left(\int_{d_1}^{d_1+\xi_c^*} + \int_{d_2}^{d_2+\xi_a^*-\xi_c^*} + \int_{d_3}^{d_3+\xi_b^*-\xi_a^*} + \int_{d_4}^{\infty}\right) S_X(x)\mathrm{d}x,$$

which is a real-valued continuous function of (d_1, d_2, d_3, d_4). Thus $\{P_I : I \text{ has expression } (4.50)\} = [P_{I^m_{\xi_c^*,\xi_a^*,\xi_b^*}}, P_{I^M_{\xi_c^*,\xi_a^*,\xi_b^*}}]$. By (4.34), we have $P_{I^m_{\xi_c^*,\xi_a^*,\xi_b^*}} \le P_{\xi_c^*,\xi_a^*,\xi_b^*} \le P_{I^M_{\xi_c^*,\xi_a^*,\xi_b^*}}$ and $\xi_c^* + L_3 \le P_{\xi_c^*,\xi_a^*,\xi_b^*} \le \xi_a^* - a + L_1$. Therefore, there exists $I^* \in \mathcal{I}_2$ such that $I^*(c) = \xi_c^*$, $I^*(a) = \xi_a^*$, $I^*(b) = \xi_b^*$, and $P_{I^*} = P_{\xi_c^*,\xi_a^*,\xi_b^*}$. For any $I \in \mathcal{I}_2$, denote $\xi_c = I(c)$, $\xi_a = I(a)$, and $\xi_b = I(b)$, then

$$\begin{aligned} V(I^*) &= \lambda a + (2\lambda - 1)P_{\xi_c^*,\xi_a^*,\xi_b^*} - \lambda \xi_a^* + (1-\lambda)\xi_b^* = w(\xi_c^*, \xi_a^*, \xi_b^*) \\ &= w(\xi_c^*, \xi_a^*(\xi_c^*), \xi_b^*(\xi_c^*, \xi_a^*(\xi_c^*))) = w_2(\xi_c^*, \xi_a^*(\xi_c^*)) = w_1(\xi_c^*) = \min_{\xi \in \Xi_c} w_1(\xi) \\ &\le w_1(\xi_c) = \min_{\xi \in \Xi_{a,\xi_c}} w_2(\xi_c, \xi) \le w_2(\xi_c, \xi_a) = \min_{\xi \in \Xi_{b,\xi_c,\xi_a}} w(\xi_c, \xi_a, \xi) \\ &\le w(\xi_c, \xi_a, \xi_b) = \lambda a + (2\lambda - 1)P_{\xi_c,\xi_a,\xi_b} - \lambda\xi_a + (1-\lambda)\xi_b \le V(I), \end{aligned}$$

where the last inequality is from the proof of Lemma 4.3.6. Therefore, a contract I^* of the form (4.36) for some $(d_1, d_2, d_3, d_4) \in [0,\ c - \xi_c^*] \times [c,\ a - \xi_a^* + \xi_c^*] \times [a,\ b - \xi_b^* + \xi_a^*] \times [b,\ \infty]$, satisfying $I^*(c) = \xi_c^*$, $I^*(a) = \xi_a^*$, $I^*(b) = \xi_b^*$, and $P_{I^*} = P_{\xi_c^*,\xi_a^*,\xi_b^*}$, is an optimal solution to Problem (4.23). ■

Proof of Corollary 4.3.9. Suppose $a < b$ and $0 \le \lambda < 1/2$. By (4.27) and (4.28), we have that $\xi_c \in \Xi_c$ is equivalent to $h_1(\xi_c) \ge L_3$ and $h_2(\xi_c) \le L_1 - c$. Note that (4.6) implies $h_2(c - v_\theta) \le L_1 - c$ while (4.7) implies $h_1(v_\theta) \ge L_3$.

(a) Assume $h_2(v_\theta) \le L_1 - c$. Note that $h_1(v_\theta) \ge L_3$, thus we have $v_\theta \in \Xi_c$. By Lemma 4.3.7(a), we get $\xi_c^* = v_\theta$. It follows that $\xi_a^* = \xi_a^*(v_\theta) = \sup\{\xi_a \in \Xi_{a,v_\theta} : A_{v_\theta}(\xi_a) < v_\theta + L_3\}$, $\xi_b^* = \xi_b^*(v_\theta, \xi_a^*) = \xi_b^m(v_\theta, \xi_a^*)$, and $P_{\xi_c^*,\xi_a^*,\xi_b^*} = (L_1 - a + \xi_a^*) \wedge P_{I^M_{\xi_c^*,\xi_a^*,\xi_b^*}}$.

(i) If $L_3 \le h_4(v_\theta)$, note that $h_4(v_\theta) < h_5(v_\theta)$, thus $v_\theta + L_3 \le A^m_{v_\theta}(v_\theta + a - L_1 + L_3) < A_{v_\theta}(v_\theta + a - L_1 + L_3)$. From **Case a.i** in the proof of Lemma 4.3.7, we have $\xi_b^* = \xi_a^* = \xi_a^m(v_\theta)$. We will specify the value of $\xi_a^m(v_\theta)$. Since $(v_\theta + a - L_1 + L_3) + L_1 - a \le A^m_{v_\theta}(v_\theta + a - L_1 + L_3)$ from $L_3 \le h_4(v_\theta)$, $(v_\theta + a - c) + L_1 - a \ge A^m_{v_\theta}(v_\theta + a - c)$ from $h_2(v_\theta) \le L_1 - c$, and $A^m_{v_\theta}(\xi_a)$ is continuous and strictly increasing in $\xi_a \in \Xi_{a,v_\theta}$, there exists $\xi_{a,0} \in [v_\theta + L_3 + a - L_1, v_\theta + a - c]$, which is the unique solution to the equation of $\xi_{a,0} + L_1 - a = A^m_{v_\theta}(\xi_{a,0})$. Hence, $\xi_{a,0}$ satisfies (4.30) for $\xi_c = v_\theta$. Meanwhile, $\xi_{a,0}$ satisfies (4.29) for $\xi_c = v_\theta$ because $v_\theta + L_3 \le \xi_{a,0} + L_1 - a = A^m_{v_\theta}(\xi_{a,0}) < A^M_{v_\theta}(\xi_{a,0})$. Thus, $\xi_{a,0} \in \Xi_{a,v_\theta}$. For any $\xi_a < \xi_{a,0}$, since $\xi_a - A^m_{v_\theta}(\xi_a)$ is strictly increasing, we know that $\xi_a - A^m_{v_\theta}(\xi_a) < \xi_{a,0} - A^m_{v_\theta}(\xi_{a,0}) = a - L_3$. It implies that (4.30) with $\xi_c = v_\theta$ is not satisfied by ξ_a, and then $\xi_a \notin \Xi_{a,v_\theta}$. Therefore, $\xi_a^m(v_\theta) = \xi_{a,0}$. It follows that $\xi_b^* = \xi_a^* = \xi_{a,0}$ and $P_{\xi_c^*,\xi_a^*,\xi_b^*} = (\xi_{a,0} + L_1 - a) \wedge P_{I^M_{v_\theta,\xi_{a,0},\xi_{a,0}}} = (\xi_{a,0} + L_1 - a) \wedge A_{v_\theta}(\xi_{a,0}) = \xi_{a,0} + L_1 - a$. Hence, $I^*(x) = (x - c + v_\theta)^+ - (x - c)^+ + (x - (a - \xi_{a,0} + v_\theta))^+ - (x - a)^+$ since it is easy to check that $I^*(c) = v_\theta$, $I^*(a) = I^*(b) = \xi_{a,0}$, and $P_{I^*} = A^m_{v_\theta}(\xi_{a,0}) = \xi_{a,0} + L_1 - a$. Thus, I^* is the optimal contract by Theorem 4.3.8.

(ii) If $h_4(v_\theta) < L_3 \le h_5(v_\theta)$, which means $A^m_{v_\theta}(v_\theta + a - L_1 + L_3) < (v_\theta + L_3 + a - L_1) + L_1 - a = v_\theta + L_3 \le A_{v_\theta}(v_\theta + a - L_1 + L_3)$, then $v_\theta + L_3 + a - L_1$ satisfies (4.29) and (4.30) for $\xi_c = v_\theta$. It implies $v_\theta + L_3 + a - L_1 \in \Xi_{a,v_\theta}$, where $\Xi_{a,v_\theta} \subset [v_\theta + L_3 + a - L_1, a]$ by its definition, and thus, $\xi_a^m(v_\theta) = v_\theta + L_3 + a - L_1$. From **Case a.i** in the proof of Lemma 4.3.7, we have $\xi_b^* = \xi_a^* = \xi_a^m(v_\theta) = v_\theta + L_3 + a - L_1$ and $P_{\xi_c^*,\xi_a^*,\xi_b^*} = (\xi_a^* + L_1 - a) \wedge P_{I^M_{v_\theta,\xi_a^*,\xi_a^*}} = (\xi_a^* + L_1 - a) \wedge A_{v_\theta}(v_\theta + a - L_1 + L_3) = v_\theta + L_3$. As a function of $(d_1, d_2, d_3) \in [0, c - v_\theta] \times [c, L_1 - L_3] \times [b, \infty]$,

$$P_I = P_I(d_1, d_2, d_3) = (1 + \theta)\left(\int_{d_1}^{d_1 + v_\theta} + \int_{d_2}^{d_2 + a + L_3 - L_1} + \int_{d_3}^{\infty}\right) S_X(x)\mathrm{d}x$$

can take all values on $[P_I(c - v_\theta, L_1 - L_3, \infty), P_I(0, c, b)]$. Since $P_I(c - v_\theta, L_1 - L_3, \infty) = h_4(v_\theta) + v_\theta < L_3 + v_\theta \le h_5(v_\theta) + v_\theta = P_I(0, c, b)$, there exists $(d_1^*, d_2^*, d_3^*) \in [0, c - v_\theta] \times [c, L_1 - L_3] \times [b, \infty]$ such that $P_I(d_1^*, d_2^*, d_3^*) = v_\theta + L_3$. Therefore,

$$I^*(x) = (x - d_1^*)^+ - (x - d_1^* - v_\theta)^+ + (x - d_2^*)^+ - (x - d_2^* - (a - L_1 + L_3))^+ + (x - d_3^*)^+$$

because it satisfies $I^*(x) = \xi_x^*$ for $x = c, a, b$ and $P_{I^*} = P_I(d_1^*, d_2^*, d_3^*) = v_\theta + L_3$.

(iii) If $h_5(v_\theta) < L_3$, by the arguments in **Case a.ii** in the proof of Lemma 4.3.7, we know that there exists $\xi_{a,1} \in [v_\theta + L_3 + a - L_1, v_\theta + b - c]$ such that $A_{v_\theta}(\xi_{a,1}) = v_\theta + L_3$ and $(\xi_c^*, \xi_a^*, \xi_b^*) = (v_\theta, (v_\theta + a - c) \wedge \xi_{a,1}, \xi_{a,1})$. It implies that $P_{\xi_c^*,\xi_a^*,\xi_b^*} = (\xi_a^* - a + L_1) \wedge P_{I^M_{\xi_c^*,\xi_a^*,\xi_b^*}} = (\xi_a^* - a + L_1) \wedge A_{v_\theta}(\xi_{a,1}) = v_\theta + L_3$. Hence, $I^*(x) = x - (x - v_\theta)^+ + (x - c)^+ - (x - (c + \xi_{a,1} - v_\theta))^+ + (x - b)^+$ since it easy to check that $I^*(x) = \xi_x^*$ for $x = c, a, b$ and $P_{I^*} = A_{v_\theta}(\xi_{a,1}) = v_\theta + L_3$.

(b) Assume $h_2(v_\theta) > L_1 - c$. Note that $h_2(c-v_\theta) \le L_1 - c$ and h_2 is continuous and monotone on $[v_\theta \wedge (c-v_\theta), v_\theta \vee (c-v_\theta)]$, thus the equation $h_2(\xi_c) = L_1 - c$ has solutions on $[v_\theta \wedge (c-v_\theta), v_\theta \vee (c-v_\theta)]$. Denote

$$\xi_{L_1-c,\,h_2} = \inf\left\{\xi_c \in [v_\theta \wedge (c-v_\theta), v_\theta \vee (c-v_\theta)] : \, h_2(\xi_c) = L_1 - c\right\}. \tag{4.51}$$

Notice that $L_3 \le L_1 - c = h_2(\xi_{L_1-c,\,h_2}) < h_1(\xi_{L_1-c,\,h_2})$ implies that (4.27) and (4.28) are satisfied by $\xi_{L_1-c,\,h_2}$ and thus $\xi_{L_1-c,\,h_2} \in \Xi_c$. Suppose $v_\theta < c - v_\theta$, then $v_\theta \le \xi_{L_1-c,\,h_2} \le c - v_\theta$. For any $\xi_c < \xi_{L_1-c,\,h_2}$, we have $h_2(\xi_c) > L_1 - c$ because h_2 is decreasing on $[0, c-v_\theta]$. It implies that $\xi_c \notin \Xi_c$ because it does not satisfy (4.28). Thus $\xi_{L_1-c,\,h_2} = \xi_c^m$ and moreover, $v_\theta \le \xi_{L_1-c,\,h_2} = \xi_c^m \le \xi_c^M$. By Lemma 4.3.7(a), we have $\xi_c^* = \xi_c^m \vee (v_\theta \wedge \xi_c^M) = \xi_{L_1-c,\,h_2}$. In the other case of $v_\theta \ge c - v_\theta$, we have $h_2(\xi_c) > h_2(\xi_{L_1-c,\,h_2}) = L_1 - c$ for any $\xi_c > \xi_{L_1-c,\,h_2}$ because h_2 is strictly increasing on $[c-v_\theta, c]$. It implies that $\xi_c \notin \Xi_c$ because it does not satisfy (4.28). Thus $\xi_c^M = \xi_{L_1-c,\,h_2} \le v_\theta$. By Lemma 4.3.7(a), we have $\xi_c^* = \xi_c^m \vee (v_\theta \wedge \xi_c^M) = \xi_{L_1-c,\,h_2}$. Therefore, in both of the two cases, $\xi_c^* = \xi_{L_1-c,\,h_2}$. Note that the equation $h_2(\xi_{L_1-c,\,h_2}) = L_1 - c$ can be rewritten as $A^m_{\xi_c^*}(\xi_c^* + a - c) = (\xi_c^* + a - c) + L_1 - a$. Since the function $\xi_a - A^m_{\xi_c^*}(\xi_c)$ is strictly increasing in ξ_a, for any $\xi_a < \xi_c^* + a - c$, we have $\xi_a - A^m_{\xi_c^*}(\xi_c) < (\xi_c^* + a - c) - A^m_{\xi_c^*}(\xi_c^* + a - c) = a - L_1$, which means that (4.30) with $\xi_c = \xi_c^*$ are not satisfied by ξ_a. Thus, $\Xi_{a,\xi_c^*} = \{\xi_c^* + a - c\}$ is a single point set. It is easy to check that $\xi_b^* = \xi_b^m(\xi_c^*, \xi_a^*) = \xi_a^* = \xi_c^* + a - c$ and $P_{\xi_c^*,\xi_a^*,\xi_b^*} = (\xi_a^* - a + L_1) \wedge P_{I^M_{\xi_c^*,\xi_a^*,\xi_b^*}} = \xi_a^* - a + L_1$, where $P_{I^M_{\xi_c^*,\xi_a^*,\xi_b^*}} = A_{\xi_c^*}(\xi_a^*) \ge A^m_{\xi_c^*}(\xi_a^*) = h_2(\xi_c^*) + \xi_c^* = L_1 - c + \xi_c^* = \xi_a^* - a + L_1$. The contract $I^*(x) = (x - c + \xi_{L_1-c,h_2})^+ - (x-a)^+$ is the optimal one because it satisfies $I^*(x) = \xi_x^*$ for $x = c, a, b$ and $P_{I^*} = h_2(\xi_c^*) - \xi_c^* = \xi_c^* - c + L_1$. ■

Proof of Corollary 4.3.10. Suppose $a < b$ and $1/2 < \lambda \le 1$. By Lemma 4.3.7(b), we have that $\xi_c^* = \xi_{L_3,h_2} \vee \xi_{L_3,h_3}$, $\xi_a^* = \xi_c^* + a - c$, $\xi_b^* = \xi_b^m(\xi_a^*, \xi_c^*)$ and $P_{\xi_c^*,\xi_a^*,\xi_b^*} = (\xi_c^* + L_3) \vee P_{I^m_{\xi_c^*,\xi_a^*,\xi_b^*}}$, where $\xi_{L_3,h_2} = \sup\{\xi_c \in [0, c-v_\theta] : \, h_2(\xi_c) \ge L_3\}$ and $\xi_{L_3,h_3} = \sup\{\xi_c \in [0, v_\theta] : \, h_3(\xi_c) \le L_3\}$. Note that $h_2(c - v_\theta) \le h_2(0) < h_3(0) \le h_3(v_\theta)$.

(a) If $L_3 \le h_2(0)$, from **Case b.i** in the proof of Lemma 4.3.7, we have $\xi_c^* = \xi_{L_3,h_2}$, $\xi_a^* = \xi_c^* + a - c$, and $h_2(\xi_c^*) \ge L_3$, where $h_2(\xi_c^*) = A^m_{\xi_c^*}(\xi_c^* + a - c) - \xi_c^* = A^m_{\xi_c^*}(\xi_a^*) - \xi_c^*$. Since $h_3(\xi_c^*) \ge h_2(\xi_c^*) \ge L_3$, namely $A_{\xi_c^*}(\xi_a^*) \ge \xi_c^* + L_3$, from **Case b.2** in the proof of Lemma 4.3.7, we have $\xi_b^* = \xi_a^* = \xi_c^* + a - c$. Then $P_{\xi_c^*,\xi_a^*,\xi_b^*} = (\xi_c^* + L_3) \vee P_{I^m_{\xi_c^*,\xi_a^*,\xi_b^*}} = (\xi_c^* + L_3) \vee A^m_{\xi_c^*}(\xi_a^*) = A^m_{\xi_c^*}(\xi_a^*)$. Consider the contract $I^*(x) = (x - c + \xi_c^*)^+ - (x-a)^+$, it is easy to check that $I^*(x) = \xi_x^*$ for $x = c, a, b$ and $P_{I^*} = (1+\theta)\int_{c-\xi_{L_3,h_2}}^{a} S_X(x)\mathrm{d}x = A^m_{\xi_c^*}(\xi_a^*)$. Thus, I^* is the optimal contract by Theorem 4.3.8.

(b) If $h_2(0) < L_3 < h_3(0)$, from **Case b.iii** in the proof of Lemma 4.3.7, we have $\xi_c^* = 0$ and $\xi_a^* = a - c$. Since $h_3(0) > L_3$, namely, $A_0(a-c) = h_3(0) > L_3$, from **Case**

b.2 in the proof of Lemma 4.3.7, we have $\xi_b^* = \xi_a^* = a - c$ and moreover $P_{\xi_c^*,\xi_a^*,\xi_b^*} = (\xi_c^* + L_3) \vee P_{I^m_{\xi_c^*,\xi_a^*,\xi_b^*}} = L_3 \vee A_0^m(a-c) = L_3 \vee h_2(0) = L_3$. As a function of d, for any $d \in [b,\infty]$, $P_I = P_I(d) = (1+\theta)\left(\int_c^a + \int_d^\infty\right) S_X(x)\mathrm{d}x$ is continuous and decreasing in d. Note that $P_I(\infty) = h_2(0) < L_3 < h_3(0) = P_I(b)$, thus there exists $d^* \in [b,\infty]$ such that $P_I(d^*) = L_3$. The contract $I^*(x) = (x-c)^+ - (x-a)^+ + (x-d^*)^+$ is optimal because it satisfies $I^*(x) = \xi_x^*$ for $x = c, a, b$ and $P_{I^*} = P_I(d^*) = L_3$.

(c) If $h_3(0) \le L_3$, from **Case b.ii** in the proof of Lemma 4.3.7, we have $\xi_c^* = \xi_{L_3,h_3}$, $\xi_a^* = \xi_{L_3,h_3} + a - c$, and $h_3(\xi_c^*) \le L_3$. Since $h_3(\xi_c^*) = A_{\xi_c^*}(\xi_c^* + a - c) - \xi_c^* = A_{\xi_c^*}(\xi_a^*) - \xi_c^*$, we have $A_{\xi_c^*}(\xi_a^*) = h_3(\xi_c^*) + \xi_c^* \le \xi_c^* + L_3$. From **Case b.1** in the proof of in Lemma 4.3.7, we have that ξ_b^* is the solution to the equation of $\xi_c^* + L_3 = P_{I^M_{\xi_c^*,\xi_a^*,\xi_b^*}}$ and it can be checked that $A_{\xi_c^*}(\xi_b^*) = P_{I^M_{\xi_c^*,\xi_a^*,\xi_b^*}} = (\xi_c^* + L_3) \vee P_{I^m_{\xi_c^*,\xi_a^*,\xi_b^*}} = P_{\xi_c^*,\xi_a^*,\xi_b^*}$. Therefore, $I^*(x) = x - (x - \xi_{L_3,h_3})^+ + (x-c)^+ - (x - (c + \xi_b^* - \xi_{L_3,h_3}))^+ + (x-b)^+$ because it satisfies $I^*(x) = \xi_x^*$ for $x = c, a, b$ and $P_{I^*} = A_{\xi_c^*}(\xi_b^*) = \xi_c^* + L_3$. ∎

Proof of Corollary 4.3.11. Suppose $b < a$ and $0 \le \lambda < 1/2$. By Lemma 4.3.7(c), we have $\xi_c^* = \xi_c^m \vee (v_\theta \wedge \xi_c^M)$, $\xi_b^* = \xi_b^m(\xi_c^*)$, $\xi_a^* = \xi_b^* + a - b$, and $P_{\xi_c^*,\xi_a^*,\xi_b^*} = (\xi_b^* - b + L_1) \wedge B^M_{\xi_c^*}(\xi_b^*)$.

(a) Assume $h_2(v_\theta) \le L_1 - c$, which means $v_\theta \in \Xi_c$, thus $\xi_c^* = v_\theta$.

(i) If $(b + L_3 - L_1)^+ + L_1 - b < h_6(v_\theta)$, which is equivalent to $v_\theta + (b+L_3-L_1)^+ + L_1 - b < B^m_{v_\theta}(v_\theta + (b + L_3 - L_1)^+)$, note that $(v_\theta + b - c) + L_1 - b = v_\theta + L_1 - c \ge v_\theta + h_2(v_\theta) = B^m_{v_\theta}(v_\theta + b - c)$ and $\xi_b - B^m_{v_\theta}(\xi_b)$ is continuous and strictly increasing in ξ_b, thus there exists $\xi_{b,0} \in [v_\theta + (b + L_3 - L_1)^+, v_\theta + b - c]$ such that $\xi_{b,0} + L_1 - b = B^m_{v_\theta}(\xi_{b,0})$, namely $\xi_{b,0}$ satisfies (4.32) for $\xi_c = v_\theta$. Moreover, $v_\theta + L_3 \le \xi_{b,0} + L_1 - b = B^m_{v_\theta}(\xi_{b,0}) \le B^M_{v_\theta}(\xi_{b,0})$ implies that $\xi_{b,0}$ satisfies (4.31) for $\xi_c = v_\theta$. Thus, $\xi_{b,0} \in \Xi_{b,v_\theta}$. For any $\xi_b < \xi_{b,0}$, $\xi_b \notin \Xi_{b,v_\theta}$ because it does not satisfy (4.32) from $\xi_b - B^m_{v_\theta}(\xi_b) < \xi_{b,0} - B^m_{v_\theta}(\xi_{b,0}) = b - L_1$. Thus, $\xi_b^m(v_\theta) = \xi_{b,0}$ and moreover, $P_{\xi_c^*,\xi_a^*,\xi_b^*} = (\xi_{b,0} - b + L_1) \wedge B^M_{v_\theta}(\xi_{b,0}) = \xi_{b,0} - b + L_1$. Then $I^*(x) = (x - c + v_\theta)^+ - (x-c)^+ + (x - (b - \xi_{b,0} + v_\theta))^+ - (x-a)^+$ because it satisfies $I^*(x) = \xi_x^*$ for $x = c, a, b$ and $P_{I^*} = B^m_{v_\theta}(\xi_{b,0}) = \xi_{b,0} + L_1 - b$.

(ii) If $h_6(v_\theta) \le (b+L_3-L_1)^+ + L_1 - b < h_7(v_\theta)$, which is equivalent to $B^m_{v_\theta}(v_\theta + (b+L_3 - L_1)^+) \le v_\theta + (b+L_3-L_1)^+ + L_1 - b < B^M_{v_\theta}(v_\theta + (b+L_3-L_1)^+)$, note that $\xi_c^* + L_3 = v_\theta + L_3 \le v_\theta + (b+L_3-L_1)^+ + L_1 - b$, thus $v_\theta + (b+L_3-L_1)^+$ satisfies (4.31) and (4.32) for $\xi_c = v_\theta$. Since $\Xi_{b,v_\theta} \subset [v_\theta + (b + L_3 - L_1)^+, b]$, we have $\xi_b^* = \xi_b^m(v_\theta) = v_\theta + (b + L_3 - L_1)^+ \in \Xi_{b,v_\theta}$. Moreover, $P_{\xi_c^*,\xi_a^*,\xi_b^*} = B^M_{v_\theta}(\xi_b^*) \wedge (\xi_b^* + L_1 - b) = v_\theta + (b + L_3 - L_1)^+ + L_1 - b$. As a function of $(d_1, d_2) \in [0, c - v_\theta] \times [a, \infty]$,

$$P_I = P_I(d_1,d_2) = (1+\theta)\left(\int_{d_1}^{d_1+v_\theta} + \int_c^{c+(b+L_3-L_1)^+} + \int_b^a + \int_{d_2}^\infty\right) S_X(x)\mathrm{d}x$$

can take all the values on $[P_I(c-v_\theta,\infty),\ P_I(0,a)]$. Note that $P_I(c-v_\theta,\infty)=v_\theta+h_6(v_\theta)\le v_\theta+(b+L_3-L_1)^+ +L_1-b<v_\theta+h_7(v_\theta)=P_I(0,a)$, thus there exist $d_1^*\in[0,c-v_\theta]$ and $d_2^*\in[a,\infty]$ such that $P_I(d_1^*,d_2^*)=v_\theta+(b+L_3-L_1)^+ +L_1-b$. Hence,

$$\begin{aligned}I^*(x)=&(x-d_1^*)^+-(x-d_1^*-v_\theta)^+ +(x-c)^+\\&-(x-c-(b+L_3-L_1)^+)^+ +(x-b)^+-(x-a)^+ +(x-d_2^*)^+\end{aligned}$$

because it satisfies $I^*(x)=\xi_x^*$ for $x=c,a,b$ and $P_{I^*}=P_I(d_1^*,d_2^*)=v_\theta+L_3\vee(L_1-b)$.

(iii) If $L_3<h_7(v_\theta)\le(b+L_3-L_1)^+ +L_1-b$, which is equivalent to $v_\theta+L_3<B^M_{v_\theta}(v_\theta+(b+L_3-L_1)^+)\le v_\theta+(b+L_3-L_1)^+ +L_1-b$, using similar arguments as in case (b), we have $\xi_b^*=\xi_b^m(\xi_c^*)=v_\theta+(b+L_3-L_1)^+$ and $P_{\xi_c^*,\xi_a^*,\xi_b^*}=B^M_{v_\theta}(\xi_b^*)\wedge(\xi_b^*+L_1-b)=B^M_{v_\theta}(\xi_b^*)$. Hence, $I^*(x)=x-(x-v_\theta)^+ +(x-c)^+-(x-c-(b+L_3-L_1)^+)^+ +(x-b)^+$ because it satisfies $I^*(x)=\xi_x^*$ for $x=c,a,b$ and $P_{I^*}=B^M_{v_\theta}(\xi_b^*)$.

(iv) If $h_7(v_\theta)\le L_3$, which is equivalent to $B^M_{v_\theta}(v_\theta+(b+L_3-L_1)^+)\le v_\theta+L_3$, using similar arguments as in **(i)**, there exists $\xi_{b,1}\in[v_\theta+(b+L_3-L_1)^+,\ v_\theta+b-c]$ such that $v_\theta+L_3=B^M_{v_\theta}(\xi_{b,1})$. Moreover, $\xi_b^*=\xi_b^m(\xi_c^*)=\xi_{b,1}$ and $P_{\xi_c^*,\xi_a^*,\xi_b^*}=(\xi_{b,1}-b+L_1)\wedge B^M_{v_\theta}(\xi_{b,1})=B^M_{v_\theta}(\xi_{b,1})=v_\theta+L_3$. Hence, $I^*(x)=x-(x-v_\theta)^+ +(x-c)^+-(x-(c+\xi_{b,1}-v_\theta))^+ +(x-b)^+$ because it satisfies $I^*(x)=\xi_x^*$, for $x=c,a,b$ and $P_{I^*}=B^M_{v_\theta}(\xi_{b,1})=v_\theta+L_3$.

(b) Assume $h_2(v_\theta)>L_1-c$. Using the same arguments for the proof of Corollary 4.3.9(b), we have $v_\theta\notin\Xi_c$ and $\xi_c^*=\xi_{L_1-c,\,h_2}$ that is defined by (4.51). For any $\xi_b<\xi_c^*+b-c$, since $\xi_b-B^m_{\xi_c^*}(\xi_b)$ is continuous and strictly increasing in ξ_b, we have $\xi_b-B^m_{\xi_c^*}(\xi_b)<(\xi_c^*+b-c)-B_{\xi_c^*}(\xi_c^*+b-c)=-h_2(\xi_c^*)+b-c=b-L_1$. It implies that ξ_b does not satisfies (4.32) and thus $\xi_b\notin\Xi_{b,\xi_c^*}$. Therefore, $\Xi_{b,\xi_c^*}=\{\xi_c^*+b-c\}$ is a single point set. Moreover, $\xi_a^*=\xi_c^*+a-c$ and $P_{\xi_c^*,\xi_a^*,\xi_b^*}=(\xi_b^*-b+L_1)\wedge B^M_{\xi_c^*}(\xi_b^*)=\xi_b^*-b+L_1=\xi_c^*-c+L_1$. Hence, $I^*(x)=(x-c+\xi_c^*)^+-(x-a)^+$ because it satisfies $I^*(x)=\xi_x^*$ for $x=c,a,b$ and $P_{I^*}=B^m_{\xi_c^*}(\xi_b^*)=\xi_c^*-c+L_1$. ∎

Proof of Corollary 4.3.12. Suppose $b<a$ and $1/2<\lambda\le1$. By Lemma 4.3.7(d), we have $\xi_c^*=\xi_c^m\vee\left[(c-v_\theta)\wedge\xi_c^M\right]$, $\xi_b^*=\xi_c^*+b-c$, $\xi_a^*=\xi_c^*+a-c$, and $P_{\xi_c^*,\xi_a^*,\xi_b^*}=(\xi_c^*+L_3)\vee P_{I^m_{\xi_c^*,\xi_a^*,\xi_b^*}}=(\xi_c^*+L_3)\vee(h_2(\xi_c^*)+\xi_c^*)$. By (4.27) and (4.28), we know that $c-v_\theta\in\Xi_c$ is equivalent to $h_1(c-v_\theta)\ge L_3$ and $h_2(c-v_\theta)\le L_1-c$. Note that $h_2(c-v_\theta)\le L_1-c$ by (4.7).

(a) If $h_1(c-v_\theta)<L_3$, then $c-v_\theta\notin\Xi_c$. Furthermore, note that $h_1(v_\theta)\ge L_3$ by (4.7) and h_1 is continuous and monotone on $[v_\theta\wedge(c-v_\theta),\ v_\theta\vee(c-v_\theta)]$, thus the equation $h_1(\xi_c)=L_3$ has solutions on $[v_\theta\wedge(c-v_\theta),\ v_\theta\vee(c-v_\theta)]$. Denote

$$\xi_{L_3,h_1}=\sup\left\{\xi_c\in[v_\theta\wedge(c-v_\theta),\ v_\theta\vee(c-v_\theta)]:\ h_1(\xi_c)=L_3\right\}. \tag{4.52}$$

Then, we have $h_1(\xi_{L_3,h_1}) = L_3$. Moreover, $h_2(\xi_{L_3,h_1}) \le h_1(\xi_{L_3,h_1}) = L_3 \le L_1 - c$. Thus, $\xi_{L_3,h_1} \in \Xi_c$. Suppose $v_\theta \le \xi_{L_3,h_1} \le c - v_\theta$, since h_1 is decreasing on $[v_\theta, c - v_\theta]$, we have $h_1(\xi_c) < L_3$, for any $\xi_c > \xi_{L_3,h_1}$, namely ξ_c does not satisfy (4.27) and $\xi_c \notin \Xi_c$. It implies that $\xi_c^M = \xi_{L_3,h_1} \le c - v_\theta$ and thus $\xi_c^* = \xi_{L_3,h_1}$. Suppose $c - v_\theta \le \xi_{L_3,h_1} \le v_\theta$, since h_1 is strictly increasing on $[c - v_\theta, v_\theta]$, we have $h_1(\xi_c) < h_1(\xi_{L_3,h_1}) = L_3$ for any $\xi_c < \xi_{L_3,h_1}$, namely ξ_c does not satisfy (4.27) and $\xi_c \notin \Xi_c$. We also conclude that $\xi_c^* = \xi_c^m = \xi_{L_3,h_1}$. Moreover, $P_{\xi_c^*,\xi_a^*,\xi_b^*} = (\xi_{L_3,h_1} + L_3) \vee (h_2(\xi_{L_3,h_1}) + \xi_{L_3,h_1}) = L_3 + \xi_{L_3,h_1}$, where $h_2(\xi_{L_3,h_1}) \le h_1(\xi_{L_3,h_1}) = \xi_{L_3,h_1}$. The optimal contract is $I^*(x) = x - (x - \xi_{L_3,h_1})^+ + (x - c)^+$ because it satisfies $I^*(x) = \xi_x^*$ for $x = c, a, b$ and $P_{I^*} = \xi_{L_3,h_1} + h_1(\xi_{L_3,h_1}) = \xi_{L_3,h_1} + L_3$.

(b) If $h_1(c - v_\theta) \ge L_3$ which means $c - v_\theta \in \Xi_c$, then $\xi_c^* = c - v_\theta$, $\xi_b^* = b - v_\theta$, $\xi_a^* = a - v_\theta$, and $P_{\xi_c^*,\xi_a^*,\xi_b^*} = c - v_\theta + L_3 \vee h_2(c - v_\theta)$. As a function of $(d_1, d_2) \in [0, v_\theta] \times [a, \infty]$, $P_I = P_I(d_1, d_2) = (1+\theta)\left(\int_{d_1}^{d_1+c-v_\theta} + \int_c^a + \int_{d_2}^{\infty}\right) S_X(x)\mathrm{d}x$ is continuous and can take all the values on $[P_I(0, a), P_I(v_\theta, \infty)]$. Note that $h_1(c - v_\theta) \ge L_3$ by (4.27) and $h_1(\xi_c) \ge h_2(\xi_c)$ for all $\xi_c \in [0, c]$, then $P_I(0, a) = h_1(c - v_\theta) + c - v_\theta \ge L_3 \vee h_2(c - v_\theta) + c - v_\theta$. Together with $P_I(v_\theta, \infty) = h_2(c - v_\theta) + c - v_\theta \le L_3 \vee h_2(c - v_\theta) + c - v_\theta$, we know that there exists $(d_1^*, d_2^*) \in [0, v_\theta] \times [a, \infty]$ such that $P_I(d_1^*, d_2^*) = P_{\xi_c^*,\xi_a^*,\xi_b^*}$. The optimal contract is $I^*(x) = (x - d_1^*)^+ - (x - d_1^* - c + v_\theta)^+ + (x - c)^+ - (x - a)^+ + (x - d_2^*)^+$ because it satisfies $I^*(x) = \xi_x^*$ for $x = c, a, b$ and $P_{I^*} = P_I(d_1^*, d_2^*) = L_3 \vee h_2(c - v_\theta) + c - v_\theta$. ■

Chapter 5

Future Studies

5.1 Joint perspective reinsurance model with AVaR

In Chapter 4, we consider the joint perspective of the insurer and the reinsurer in one optimal reinsurance design problem. We assume both parties use VaR to measure their own risk. It is natural to consider the cases when AVaR is used instead of VaR.

Assume that the insurer use AVaR at risk level $0 < \alpha < S_X(0)$ while the reinsurer use AVaR at risk level $0 < \beta < S_X(0)$. The linear combination of two parties' interest with weighting coefficient $\lambda \in [0, 1]$ is

$$T(I) \triangleq \lambda \mathrm{AVaR}_\alpha(X - I(X) + P_I) + (1-\lambda)\mathrm{AVaR}_\beta(I(X) - P_I), \tag{5.1}$$

for any $I \in \mathcal{I}$. Assume that the premium P_I is determined by Wang's premium principle with distortion function g_P, i.e.

$$P_I = \int_0^\infty g_P \circ S_X(t)\,\mathrm{d}t.$$

Therefore, the optimal reinsurance problem without constraints is

$$\min_{I \in \mathcal{I}} \quad T(I). \tag{5.2}$$

Note that, the value function $T(I)$ can be simplified as follows.

$$\begin{aligned} T(I) &= \lambda\left[\mathrm{AVaR}_\alpha(X) - \mathrm{AVaR}_\alpha(I(X)) + P_I\right] + (1-\lambda)\left[\mathrm{AVaR}_\beta(I(X)) - P_I\right] \\ &= \lambda \mathrm{AVaR}_\alpha(X) - \lambda \mathrm{AVaR}_\alpha(I(X)) + (1-\lambda)\mathrm{AVaR}_\beta(I(X)) + (2\lambda - 1)P_I. \end{aligned}$$

By definition, we have

$$\mathrm{AVaR}_\alpha(I(X)) = \frac{1}{\alpha}\int_a^\infty I(x)\,\mathrm{d}F_X(x) = \int_0^\infty I(x)\,\mathrm{d}F_\alpha(x)$$

where $a = \mathrm{VaR}_\alpha(X)$ and $F_\alpha(x) = (1 - S_X(x)/\alpha)\,\mathbb{I}_{[a,\infty)}(x)$, for $x \geq 0$. Note that

$$S_\alpha(x) = 1 - F_\alpha(x) = \begin{cases} 1, & 0 \leq x < a, \\ \frac{1}{\alpha}S_X(x), & a \leq x < \infty, \end{cases}$$

can be viewed as a distorted survival function, that is $S_\alpha(x) = g_\alpha \circ S_X(x)$, where

$$g_\alpha(t) = \left(\frac{t}{\alpha}\right) \vee 1 = \begin{cases} \frac{t}{\alpha}, & 0 \leq t < \alpha, \\ 1, & \alpha \leq t \leq 1 \end{cases}$$

is a distortion function. Similarly, we can define the distortion function $g_\beta(t) = (t/b) \vee 1$, for $0 \leq t \leq 1$, where $b = \mathrm{VaR}_\beta(X)$, and the corresponding distorted survival function and probability are $S_\beta(x) = g_\beta \circ S_X(x)$ and $F_\beta(x) = 1 - S_\beta(x)$, for all $x \geq 0$. Then,

$$\mathrm{AVaR}_\beta(I(X)) = \frac{1}{\beta}\int_b^\infty I(x)\,\mathrm{d}F_X(x) = \int_0^\infty I(x)\,\mathrm{d}F_\beta(x).$$

Meanwhile, let $F_g(x) = 1 - g_P \circ S_X(x)$, for $x \geq 0$, then the premium is $P_I = \int_0^\infty I(x)\,\mathrm{d}F_g(x)$. Therefore,

$$\begin{aligned} T(I) &= \lambda \mathrm{AVaR}_\alpha(X) - \lambda \int_0^\infty I(x)\,\mathrm{d}F_\alpha(x) \\ &\quad + (1-\lambda)\int_0^\infty I(x)\,\mathrm{d}F_\beta(x) + (2\lambda - 1)\int_0^\infty I(x)\,\mathrm{d}F_g(x) \\ &= \lambda \mathrm{AVaR}_\alpha(X) + \int_0^\infty I(x)\,\mathrm{d}G(x) \end{aligned}$$

where

$$\begin{aligned} G(x) &= -\lambda F_\alpha(x) + (1-\lambda)F_\beta(x) + (2\lambda - 1)F_g(x) \\ &= \lambda S_\alpha(x) - (1-\lambda)S_\beta(x) + (1-2\lambda)S_g(x), \quad x \geq 0, \\ &= \lambda g_\alpha \circ S_X(x) - (1-\lambda)g_\beta \circ S_X(x) + (1-2\lambda)g_P \circ S_X(x) \\ &= (\lambda g_\alpha - (1-\lambda)g_\beta + (1-2\lambda)g_P) \circ S_X(x) \triangleq g \circ S_X(x). \end{aligned}$$

is a combination of three distorted survival functions. Since $\lim_{x\to\infty} I(x)G(x) = 0$, we get

$$T(I) = \lambda \mathrm{AVaR}_\alpha(X) - \int_0^\infty G(x)I'(x)\,\mathrm{d}x.$$

It follows that solving Problem (5.2) is equivalent to solving

$$\max_{I\in\mathcal{I}} \int_0^\infty G(x)I'(x)\,\mathrm{d}x. \tag{5.3}$$

Denote $G^+ \triangleq \{x \geq 0 : G(x) > 0\}$, $G^- \triangleq \{x \geq 0 : G(x) < 0\}$ and $G^0 \triangleq \{x \geq 0 : G(x) = 0\}$.

Theorem 5.1.1 *The optimal reinsurance policy for Problem (5.2) has form*

$$I^*(x) = \int_0^x \mathbb{I}_{\{G^+\}}(t) + k(t)\mathbb{I}_{\{G^0\}}(t)\,\mathrm{d}t, \quad x \geq 0, \tag{5.4}$$

where $k(t)$ is an arbitrary function such that $0 \leq k(t) \leq 1$.

Proof. For an arbitrary reinsurance policy $I \in \mathcal{I}$, we know that its right derivative $I'(x) \in [0,1]$. Suppose $I^* \in \mathcal{I}$ has form (5.4), then

$$\begin{aligned}\int_0^\infty G(x)I'(x)\,\mathrm{d}x &= \int_{G^+} G(x)I'(x)\,\mathrm{d}x + \int_{G^-\cup G^0} G(x)I'(x)\,\mathrm{d}x \\ &\geq \int_{G^+} G(x)\,\mathrm{d}x = \int_{G^+} G(x)\,\mathrm{d}x + \int_{G^+0} k(x)G(x)\,\mathrm{d}x \\ &= \int_0^\infty G(x)(I^*)'(x)\,\mathrm{d}x.\end{aligned}$$

Therefore, the optimal reinsurance policy has form (5.4). ■

Corollary 5.1.2 *There exists unique $t_\lambda \in (0,1)$ such that $g(t_\lambda) = 0$.*

1. *When $0 < \lambda < 1/2$, we have $t_\lambda \in [0, t_\alpha \wedge t_\beta]$, $G^+ = [\mathrm{VaR}_{t_\lambda}(X), \infty)$ and the optimal reinsurance policy is $I^*(x) = (x - \mathrm{VaR}_{t_\lambda}(X))^+$.*

2. *When $1/2 < \lambda < 1$, we have $t_\lambda \in [t_\alpha \vee t_\beta, \alpha \vee \beta]$, $G^+ = [0, \mathrm{VaR}_{t_\lambda}(X)]$ and the optimal reinsurance policy is $I^*(x) = x - (x - \mathrm{VaR}_{t_\lambda}(X))^+$.*

Proof. Assume $\alpha < \beta$ and $1/2 < \lambda < 1$. Then $a > b$, and

$$\begin{aligned} g(t) &= \lambda g_\alpha(t) - (1-\lambda) g_\beta(t) + (1-2\lambda) g_P(t) \\ &= \begin{cases} \frac{\lambda}{\alpha} t - \frac{1-\lambda}{\beta} t + (1-2\lambda) g_P(t), & 0 \le t \le \alpha, \\ 1 - \frac{1-\lambda}{\beta} t + (1-2\lambda) g_P(t), & \alpha \le t \le \beta, \\ (1-2\lambda) g_P(t), & \beta \le t \le 1. \end{cases} \end{aligned}$$

Denote t_α such that $\alpha g_P(t_\alpha) = t_\alpha$, and t_β such that $\beta g_P(t_\beta) = t_\beta$. Then the function $\frac{t}{\alpha} \vee 1 - g_P(t) \le 0$ for $0 \le t \le t_\alpha$, and positive otherwise; while the function $\frac{t}{\beta} \vee 1 - g_P(t) \le 0$ for $0 \le t \le t_\beta$, and positive otherwise. It follows that

$$\begin{aligned} g(t) &= \lambda \left[\left(\frac{t}{\alpha} \right) \vee 1 - g_P(t) \right] - (1-\lambda) \left[\left(\frac{t}{\beta} \right) \vee 1 - g_P(t) \right] \\ &= \begin{cases} \lambda \left(\frac{t}{\alpha} - g_P(t) \right) - (1-\lambda) \left(\frac{t}{\beta} - g_P(t) \right), & 0 \le t \le t_\alpha, \\ \lambda \left(\frac{t}{\alpha} \vee 1 - g_P(t) \right) - (1-\lambda) \left(\frac{t}{\beta} - g_P(t) \right), & t_\alpha \le t \le \beta, \\ (2\lambda - 1)\left(1 - g_P(t)\right), & \beta \le t \le 1. \end{cases} \end{aligned}$$

It is easy to check that $g(t)$ is positive for $t_\alpha \le t \le t_\beta$ and $\beta \le t \le 1$.

For $t \in [t_\beta, \beta]$, it is known that $g(t_\beta) = \lambda \left(\frac{t_\beta}{\alpha} \vee 1 - g_P(t_\beta) \right) > 0$, $g(\beta) = (2\lambda - 1)\left(1 - g_P(\beta)\right) > 0$, and $g''(t) = -(2\lambda - 1) g''_P(t) \ge 0$. That is $g(t)$ is convex on $[t_\beta, \beta]$. Since $\lim_{t \to \beta} g'(t) = -\frac{1-\lambda}{\beta} + (1-2\lambda) g'_P(\beta) < 0$, it is easy to conclude that $g(t) \ge 0$ on $[t_\beta, \beta]$.

For $t \in [0, t_\alpha]$, g is convex because $g''(t) = -(2\lambda - 1) g''_P(t) \ge 0$. Note that $g(0) = 0$, $g(t_\alpha) = -(1-\lambda)\left(b_\alpha/\beta - g_P(t_\alpha)\right) \ge 0$ and $g'(t) = \frac{\lambda}{\alpha} - \frac{1-\lambda}{\beta} + (1-2\lambda) g'_P(t)$. It follows that

$$\begin{aligned} \lim_{x \downarrow 0} g'(t) &= -\infty, \\ \lim_{x \uparrow t_\alpha} g'(t) &= \frac{\lambda}{\alpha} - \frac{1-\lambda}{\beta} + (1-2\lambda) g'_P(t_\alpha) \\ &\ge \frac{\lambda}{\alpha} - \frac{1-\lambda}{\beta} + (1-2\lambda) \frac{1}{\alpha} = (1-\lambda) \left(\frac{1}{\alpha} - \frac{1}{\beta} \right) > 0, \end{aligned}$$

where $g'_P(t_\alpha) \le g_P(t_\alpha)/t_\alpha = (t_\alpha/\alpha)/t_\alpha = 1/\alpha$. Thus, there exists unique $t_\lambda \in (0, t_\alpha)$ such that $g(t) \le 0$ for $0 \le t \le t_\lambda$ and $g(t) \ge 0$ otherwise. By solving $g(t_\lambda) = 0$, we get

$$\frac{g_P(t_\lambda)}{t_\lambda} = \frac{\lambda/\alpha - (1-\lambda)/\beta}{2\lambda - 1}.$$

In summary, $g(t) \le 0$ for $0 \le t \le t_\lambda$, and $g(t) \ge 0$ otherwise. Therefore, $G(x) = g \circ S_X(x) \ge 0$ for $0 \le x \le \mathrm{VaR}_{t_\lambda}(X)$, and $G(x) \le 0$ otherwise. Moreover, $G^+ = \{x \ge 0 : G(x) \ge 0\} = [0, \mathrm{VaR}_{t_\lambda}(X)]$ and the optimal reinsurance policy is $I^*(x) = x - (x - \mathrm{VaR}_{t_\lambda}(X))^+$.

By using the same argument, we could prove the other three cases. $\blacksquare$

Remark 5.1.1 *In Problem (5.2), Wang's premium principle has the distortion form and the AVaR is a special case of distortion risk measures. Thus, the transformation from Problem (5.2) to Problem (5.3) is a promising technique to solve more general optimization problems when both the risk measure of the insurer and the premium principle have distortion expressions.*

Problem (5.2) is a minimization problem without any constraints. However, as mentioned in the introduction section, the insurer and the reinsurer could add some requirements on the optimal reinsurance policy, such as the premium budget constraint, ruin probability constraint. If the added constraints can be expressed in the distortion form, the same technique could be applied.

5.2 Policyholder's Deficit

Motived by [Asimit et al., 2013], it is interesting to consider the reinsurance impact on the policyholder's welfare. They considered the *Expected Policyholder Deficit (EPD)* which is the difference between nominal liabilities to policyholders and liabilities that will actually be paid.

Definition 5.2.1 *Given a random risk X and the available assets c, the Expected Policyholder Deficit is defined as follows:*

$$EPD(X,c) \triangleq \mathbb{E}\left[(X-c)^+\right],$$

We make the same assumption that the insurer will set up an initial reserve according to VaR-regulation. Under the consideration of default risk faced by the insurer, the nominal liabilities to policyholders is

$$\tilde{R}(X) \triangleq X - \tilde{I}(X) = X - I(X) \wedge (VaR_\alpha(I(X)) + P_I),$$

while the actually payment is $\tilde{R}(x) \wedge VaR_\gamma(\tilde{R}(X))$, where α and γ are the risk levels for the reinsurer and insurer, respectively. Thus, the optimization problem is

$$\min_{I \in \mathcal{I}} EPD\left(\tilde{R}(X), VaR_\gamma\left(\tilde{R}(X)\right)\right). \tag{5.5}$$

Recall the definition for "Expected Shortfall":

$$\begin{aligned} ES_\gamma(X) &\triangleq \frac{1}{\gamma}\int_0^\gamma VaR_\eta(X)d\eta \\ &= VaR_\gamma(X) + \frac{1}{\gamma}\mathbb{E}\left[(X - VaR_\gamma(X))^+\right]. \end{aligned}$$

Thus, Problem 5.5 can be rewritten as

$$\min_{I \in \mathcal{I}} \gamma\left[ES_\gamma\left(\tilde{R}(X)\right) - VaR_\gamma\left(\tilde{R}(X)\right)\right].$$

Lemma 5.2.1 *Denote $a \triangleq VaR_\alpha(X)$ and $c \triangleq VaR_\gamma(X)$. For each feasible reinsurance contract $I \in \mathcal{I}$, there exists $K_I \in \mathcal{I}$ satisfying*

$$EPD\left(X - \tilde{K}_I(X), VaR_\gamma\left(X - \tilde{K}_I(X)\right)\right) \le EPD\left(X - \tilde{I}(X), VaR_\gamma\left(X - \tilde{I}(X)\right)\right)$$

and K_I has form

$$K_I(x) = x - (x - \xi_\alpha \wedge \xi_\gamma)^+ + (x - a \wedge c)^+ \\ - (x - (a \wedge c + |\xi_\alpha - \xi_\gamma|))^+ + (x - a \vee c)^+, \tag{5.6}$$

for some $(\xi_\alpha, \xi_\gamma) \in \mathbb{R}^2_+$ satisfying $\xi_\alpha \leq a$ and $\xi_\gamma \leq c$.

Proof. Select an arbitrary contract $I \in \mathcal{I}$ and denote $\xi_\alpha = I(a)$ and $\xi_\gamma = I(c)$. Note that

$$ES_\gamma(X) = VaR_\gamma(X) + \frac{1}{\gamma}\mathbb{E}\left[(X - VaR_\gamma(X))^+\right].$$

where

$$ES_\gamma(X) \triangleq \frac{1}{\gamma}\int_0^\gamma VaR_\eta(X)d\eta$$

is the "Expected Shortfall" of X at level γ. Thus,

$$EPD\left(\tilde{R}(X), VaR_\gamma\left(\tilde{R}(X)\right)\right) = \gamma\left[ES_\gamma\left(\tilde{R}(X)\right) - VaR_\gamma\left(\tilde{R}(X)\right)\right].$$

It is easy to see that

$$\begin{aligned} VaR_\gamma\left(\tilde{R}(X)\right) &= \tilde{R}\left(VaR_\gamma(X)\right) \\ &= VaR_\gamma(X) - I\left(VaR_\gamma(X)\right) \wedge \left(I(VaR_\alpha(X) + P_I\right) \\ &= c - \xi_\gamma \wedge \left(\xi_\alpha + P_I\right), \end{aligned}$$

and

$$\begin{aligned} ES_\gamma\left(\tilde{R}(X)\right) &= \frac{1}{\gamma}\int_0^\gamma VaR_\eta(\tilde{R}(X))d\eta \\ &= \frac{1}{\gamma}\int_0^\gamma \tilde{R}(VaR_\eta(X))d\eta \\ &= \frac{1}{\gamma}\int_0^\gamma VaR_\eta(X)d\eta - \frac{1}{\gamma}\int_0^\gamma \tilde{I}(VaR_\eta(X))d\eta \\ &= \frac{1}{\gamma}\int_0^\gamma VaR_\eta(X)d\eta - \frac{1}{\gamma}\int_0^\gamma I(VaR_\eta(X)) \wedge \left(I(\xi_\alpha) + P_I\right)d\eta \\ &= \frac{1}{\gamma}\int_0^\gamma VaR_\eta(X)d\eta - \frac{1}{\gamma}\int_c^\infty I(t) \wedge \left(I(\xi_\alpha) + P_I\right)dF_X(t). \end{aligned}$$

As a consequence,

$$
\begin{aligned}
& EPD\left(X - \tilde{I}(X), VaR_\gamma\left(X - \tilde{I}(X)\right)\right) \\
= \;& \gamma\left[\frac{1}{\gamma}\int_0^\gamma VaR_\eta(X)d\eta - c - \frac{1}{\gamma}\int_c^\infty I(t) \wedge (I(\xi_\alpha) + P_I)\, dF_X(t) + \xi_\gamma \wedge (\xi_\alpha + P_I)\right],
\end{aligned}
$$

and Problem 5.5 is equivalent to the following minimization problem:

$$
\min_{I \in \mathcal{I}} \left\{-\frac{1}{\gamma}\int_c^\infty I(t) \wedge (I(\xi_\alpha) + P_I)\, dF_X(t) + \xi_\gamma \wedge (\xi_\alpha + P_I)\right\},
$$

because γ and $\frac{1}{\gamma}\int_0^\gamma VaR_\eta(X)d\eta - c$ are both constants. For notation simplicity, for any $I \in \mathcal{I}$, denote

$$
H(I) \triangleq -\frac{1}{\gamma}\int_c^\infty I(t) \wedge (\xi_\alpha + P_I)\, dF_X(t) + \xi_\gamma \wedge (\xi_\alpha + P_I)
$$

We are going to construct $K_I \in \mathcal{I}$ such that $K_I(c) = \xi_\gamma$ and $K_I(a) = \xi_\alpha$.

Case 1. Suppose $\gamma \geq \alpha$ (or equivalently $VaR_\gamma(X) \leq VaR_\alpha(X)$). Then $\xi_\gamma \leq \xi_\alpha < \xi_\alpha + P_I \wedge P_{K_I}$. It is easy to see that K_I defined by the expression (5.6) satisfies $K_I(t) \geq I(t)$ for all $t \geq 0$ and $P_{K_I} \geq P_I$. It implies that

$$
\begin{aligned}
H(I) - H(K_I) &= \left[-\frac{1}{\gamma}\int_c^\infty I(t) \wedge (\xi_\alpha + P_I)\, dF_X(t) + \xi_\gamma\right] \\
&\quad - \left[-\frac{1}{\gamma}\int_c^\infty K_I(t) \wedge (\xi_\alpha + P_{K_I})\, dF_X(t) + \xi_\gamma\right] \\
&= \frac{1}{\gamma}\int_c^\infty K_I(t) \wedge (\xi_\alpha + P_{K_I}) - I(t) \wedge (\xi_\alpha + P_I)\, dF_X(t) \\
&\geq 0.
\end{aligned}
$$

The last inequality is due to the fact that for all $t \geq 0$,

$$
K_I(t) \wedge (\xi_\alpha + P_{K_I}) \geq I(t) \wedge (\xi_\alpha + P_I).
$$

Case 2. Suppose $\gamma \leq \alpha$ (or equivalently $VaR_\gamma(X) \geq VaR_\alpha(X)$). Then $\xi_\gamma \geq \xi_\alpha$. Consider contract $K_I \in \mathcal{I}$ given by the expression (5.6). We have $K_I(t) \geq I(t)$ for all $t \geq 0$ and $P_{K_I} \geq P_I$.

If $\xi_\gamma \leq \xi_\alpha \leq \xi_\alpha + P_I$, then $\xi_\alpha \leq \xi_\gamma + P_I \wedge P_{K_I}$ and

$$
\begin{aligned}
H(I) - H(K_I) &= \frac{1}{\gamma}\int_c^\infty K_I(t) \wedge (\xi_\alpha + P_{K_I}) - I(t) \wedge (\xi_\alpha + P_I)\, dF_X(t) \\
&\geq 0.
\end{aligned}
$$

If $\xi_\alpha < \xi_\alpha + P_I < \xi_\gamma$, for any $x \geq c$, we have $I(x) \geq \xi_\gamma > \xi_\alpha + P_I$,

$$\begin{aligned} H(I) &= -\frac{1}{\gamma}\int_c^\infty I(t) \wedge (\xi_\alpha + P_I)\, dF_X(t) + \xi_\alpha + P_I \\ &= -\frac{1}{\gamma}\int_c^\infty (\xi_\alpha + P_I)\, dF_X(t) + \xi_\alpha + (1+\theta)\int_0^\infty I(t) dF_X(t) \\ &= -\frac{1}{\gamma}(\xi_\alpha + P_I)\, S_X(c) + \xi_\alpha + P_I \\ &= 0. \end{aligned}$$

Meanwhile

$$\begin{aligned} H(K_I) &= -\frac{1}{\gamma}\int_c^\infty K_I(t) \wedge (\xi_\alpha + P_{K_I})\, dF_X(t) + \xi_\gamma \wedge (\xi_\alpha + P_{K_I}) \\ &= \begin{cases} -\frac{1}{\gamma}(\xi_\alpha + P_{K_I})\gamma + (\xi_\alpha + P_{K_I}), & \text{if} \quad \xi_\gamma \geq \xi_\alpha + P_{K_I}; \\ -\frac{1}{\gamma}\int_c^\infty K_I(t) \wedge (\xi_\alpha + P_{K_I})\, dF_X(t) + \xi_\gamma, & \text{if} \quad \xi_\gamma \leq \xi_\alpha + P_{K_I}; \end{cases} \\ &= \begin{cases} 0, & \text{if} \quad \xi_\gamma \geq \xi_\alpha + P_{K_I}; \\ \frac{1}{\gamma}\int_c^\infty \xi_\gamma - K_I(t) \wedge (\xi_\alpha + P_{K_I})\, dF_X(t), & \text{if} \quad \xi_\gamma \leq \xi_\alpha + P_{K_I}; \end{cases} \\ &\leq 0. \end{aligned}$$

Thus, $H(K_I) \leq H(I)$ still holds when $\gamma \leq \alpha$. Combining Case 1 and Case 2, we get the result as desired

■

Remark 5.2.1 *Lemma 5.2.1 allows us to search the optimal solution among all contracts of the form (5.6) and thus Problem 5.5 can be reduced to a finite dimension minimization problem. In our future work, we plan to determine the optimal values for ξ_α and ξ_γ in the expression (5.6) and this will lead us to the optimal reinsurance contract.*

Another possible setting is assuming the maximal available assets for underlying loss X is $VaR_\gamma(X - \tilde{I}(X) + (1+\theta_1)\mathbb{E}[X]$, where $(1+\theta_1)\mathbb{E}[X]$ is the premium paid by the policyholder to the insurer. In this case, from the policyholder's point of view, the optimization problem becomes

$$\min_{I \in \mathcal{I}} EPD\left(\tilde{R}(X), VaR_\gamma\left(\tilde{R}(X)\right) + (1+\theta_1)\mathbb{E}[X]\right).$$

Meanwhile, from the insurer's point of view, the probability of default for the insurer is worth to be investigated, i.e.

$$\min_{I\in\mathcal{I}} \mathbb{P}\left(\tilde{R}(X) > VaR_{\gamma}\left(\tilde{R}(X)\right) + (1+\theta_1)\mathbb{E}[X]\right).$$

We could adopt the same construction method for Problem 5.5 to the above two optimization problems.

5.3 Multiple Reinsurers with Counterparty Default Risk

In Section 3.2, we assume there are two available reinsurers in the market which is a more general framework than the classical one reinsurer model. As mentioned in Remark 3.2.1, the insurer could reduce the premium by buying a portfolio of reinsurance contracts from multiple reinsurers. However, multiple reinsurers can lead to multiple counterparty default risks. Suppose there are two reinsurers in the market. In this case, the insurer may face counterparty default risks from both reinsurers. We make the same VaR-regulated initial reserves assumption as in Chapter 2 for each reinsurer, that is, for reinsurance contract I_i, $i=1,2$, Reinsurer i sets up an initial reserve $\omega_i \triangleq VaR_{\alpha_i}(I_i(X))$ and charges premium P_{i,I_i} from the insurer. Thus the actual indemnity paid by the Reinsurer i, $i=1,2$ under the consideration of its default risk is

$$\tilde{I}_i(x) \triangleq I_i(x) \wedge \left(VaR_{\alpha_i}(I_i(X)) + P_{i,I_i}\right), \text{ for } i=1,2.$$

Note that, two reinsurers may have different risk attitudes, which are reflected by risk level α_i for $i=1,2$, to set up his own initial reserve ω_i. In this case, the insurer's total risk becomes

$$X - \tilde{I}_1(X) - \tilde{I}_2(X) + P_{1,I_1} + P_{2,I_2},$$

and we consider the following minimization problem

$$\min_{(I_1,I_2)\in\mathcal{D}} \rho\left(X - \tilde{I}_1(X) - \tilde{I}_2(X)) + P_{1,I_1} + P_{2,I_2}\right),$$

$$\text{such that } P_{i,I_i} \triangleq (1+\theta_i)\int_0^{\infty} g_i \circ S_{I_i(X)}(x)dx,\ i=1,2,$$

where $\mathcal{D}$ is the set of all feasible pair of reinsurance contract, $\rho(\cdot)$ is a risk measure, $\theta_i \geq 0$ is the risk loading for Reinsurer i and g_i is the distortion for Reinsurer i, $i=1,2$. We plan to investigate the cases when $\rho(\cdot) = VaR_{\beta}(\cdot)$ and $\rho(\cdot) = AVaR_{\beta}(\cdot)$ where β is the risk level chosen by the insurer.

Chapter 6

Conclusion

In this thesis, we proposed three new optimal reinsurance models to reflect different requirements from both the insurer and the reinsurer.

In Chapter 2, we consider the default risk faced by the insurer due to the possibility that the reinsurer fails to pay the entire indemnity when it exceeds the reinsurer's maximal payment ability. The maximal amount that can be paid from the reinsurer, for each feasible contract I, is equal to the premium P_I plus the initial reserve ω_I based on VaR-regulation. Under the assumption that the default risk exists, we solved a utility-based maximization problem and a VaR-based minimization problem. In the utility-based model, the optimal contract may have two deductible layers in order to reduce the default risk while keeping the premium unchanged. In the VaR-based model, the optimal contract is a limited stop-loss but it requires a lower deductible when the insurer is more conservative than the reinsurer.

In Chapter 3, the insurer is assumed to minimize his total risk exposure under convex risk measure while the premium is determined by the Wang's premium principle. This is a much more general framework than the classical model. We provide a necessary condition for the expression of the optimal solution and in two particular cases, optimal solutions are given in closed form. We also consider the case when there are two reinsurers in the reinsurance market, and show that this case can be reduced to an equivalent one-reinsurer problem.

In Chapter 4, we describe feasible reinsurance contracts that are acceptable to both an insurer and a reinsurer and explore optimal reinsurance contracts which take into account both an insurer's aims and a reinsurer's goals. The models and problems proposed in this paper are interesting in theory and applications. As showed in this chapter, solving the proposed problems and finding the optimal reinsurance contracts from the perspective of both an insurer and a reinsurer are challenging jobs. The optimal reinsurance contracts from the perspectives of both an insurer and a reinsurer are more complicated than the optimal reinsurance contracts from one party's point of view only. The models and problems proposed in this chapter can be explored further in different ways such as replacing the VaR by other risk measures and accommodating other demands of an insurer and a reinsurer in the study of optimal reinsurance designs.

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